Descriptive complexity on represented spaces

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ENS Paris-Saclay (France)

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Descriptive Set Theory

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- Mathematics: on Polish spaces (separable and completely metrizable)
- Computer Science uses other spaces:
  - higher-order functionals;
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  - ...
- Development of Descriptive Set Theory on other spaces:
  - properties of representations [Brattka, 2002 & 2004];
  - ω-continuous domains [Selivanov, 2006];
  - quasi-Polish spaces [de Brecht, 2013];
  - represented spaces [de Brecht, Pauly, 2015 ; de Brecht, Schröder, Selivanov, 2016].

Represented spaces

Definition 1 Represented space

A **represented space** is a pair (X, δ):

- X is a topological space;
- $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \mapsto X$ is a **representation** (*admissible and continuous* surjective map).

Any $p \in \mathbb{N}^{\mathbb{N}}$ with $\delta(p) = x$ is a **name** of $x \in X$.

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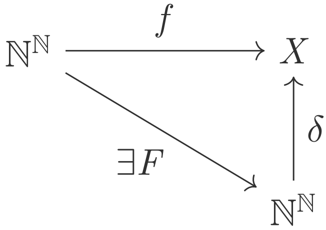

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QUESTION:

How to develop Descriptive Set Theory on represented spaces?

A motivating example

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}$$

Two competing notions of complexity:

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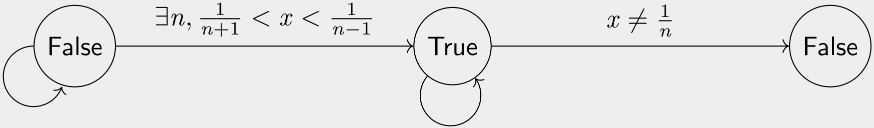
Symbolic: Deciding membership in A of an element $x \in \mathbb{R}$ (with a name).



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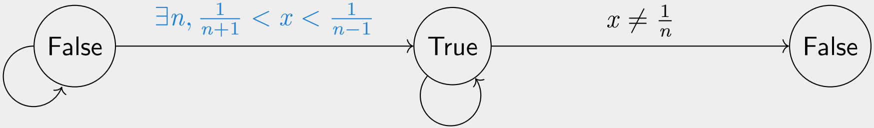

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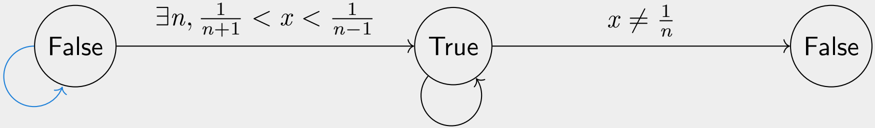

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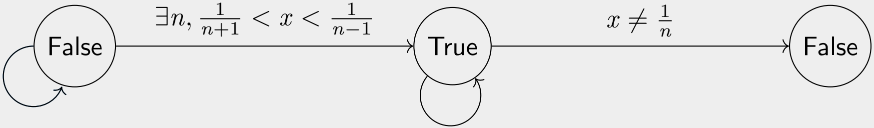

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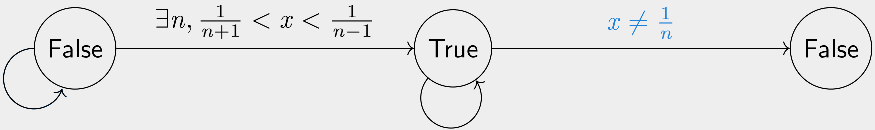

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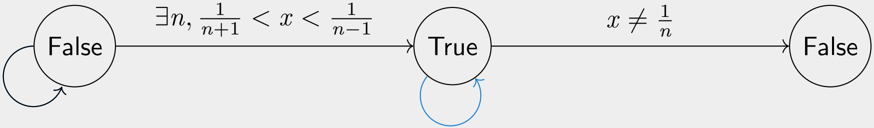

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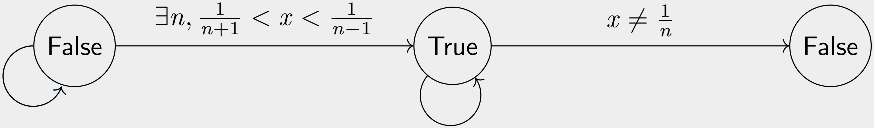

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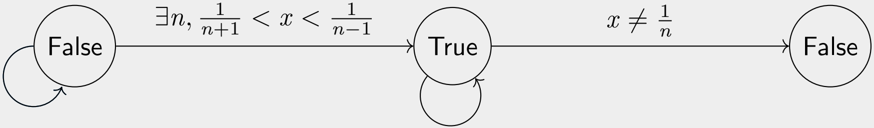

Topological:

$$A = (0, +\infty) \setminus \bigcup_{n \in \mathbb{N}^*} \left(\frac{1}{n+1}, \frac{1}{n} \right)$$

A motivating example

Symbolic descriptions and topological descriptions are equivalent on $\mathbb{R}$: for any $A \subseteq \mathbb{R}$,

A is decidable with ≤ 2 mind-changes (False $\to$ True $\to$ False)

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QUESTION:

In which spaces/for which classes of complexity is this true?

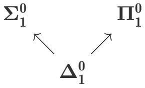
Formalization of the problem

Topological complexity

$\mathbf{\Delta}^0_1$

Borel Hierarchy

Topological complexity



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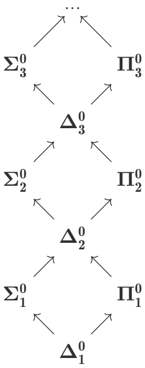

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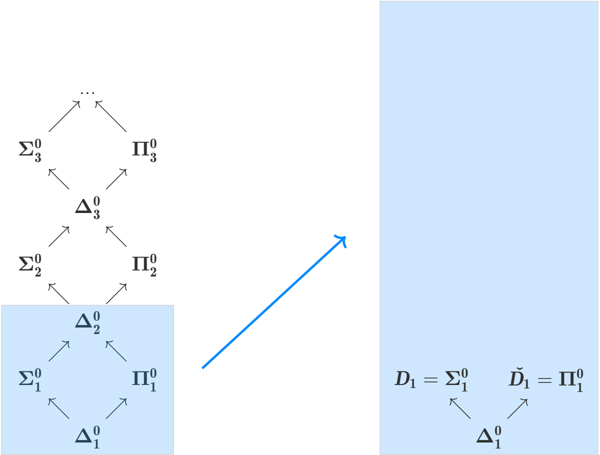
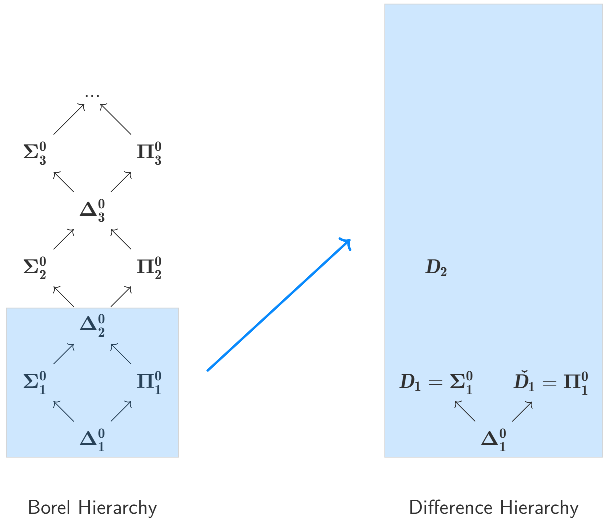


Borel Hierarchy

Difference Hierarchy

Topological complexity



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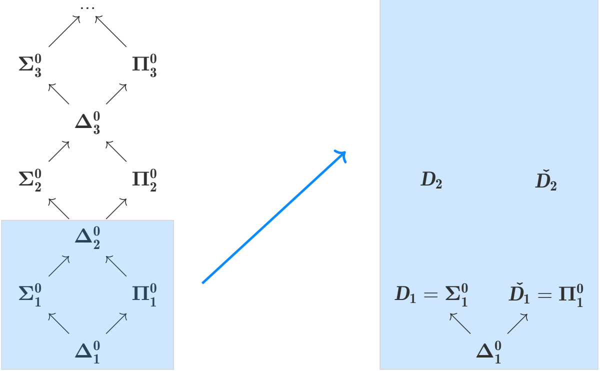

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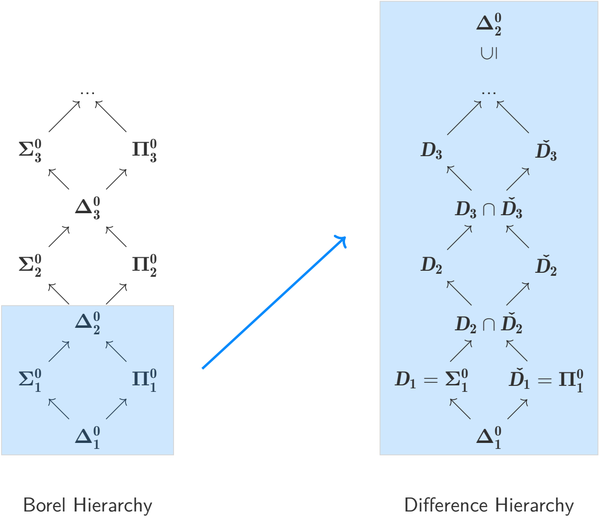

Symbolic complexity

Symbolic complexity

- On a represented space (X, δ);
- For a complexity class Γ ($\Gamma = \boldsymbol{D_2}, \boldsymbol{D_\omega}, \boldsymbol{\Pi^0_2}, \ldots$);

Definition 2 Symbolic complexity

Any $A \subseteq X$ has **symbolic complexity** Γ if

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Complexity of deciding membership in A with a relativized algorithm!

- $A \in [\boldsymbol{\Sigma^0_1}]$ iff deciding $x \in A$ is recursively enumerable (one "mind change");
- $A \in [\boldsymbol{D_2}]$ iff deciding $x \in A$ requires at most two "mind changes";
- In general: bound on the number of "mind changes" $\iff$ bound on the differences of open sets.

Symbolic and topological complexity

- Topological implies symbolic complexity: $\Gamma \subseteq [\Gamma]$ by continuity of δ;
- Equivalence for semi-decidable/open sets: $\mathbf{\Sigma^0_1} = [\mathbf{\Sigma^0_1}]$ by admissibility of δ;

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> QUESTION:
>
> **For which classes Γ/spaces do we have $\Gamma = [\Gamma]$?**

In this talk:

- A class of spaces with $\Gamma = [\Gamma]$: countably-based spaces;
- Examples of spaces with $\Gamma \neq [\Gamma]$: some coPolish spaces, spaces of open sets, etc...

Case 1: countably-based spaces

Countably-based spaces

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$$S \in \boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}(\mathrm{dom}(\delta)) \implies \varphi(S) \in \boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}(X)$$

Countably-based spaces

Theorem 3 [De Brecht, 2013], [C.,Hoyrup, 2020]

If X is countably-based, then $\Gamma = [\Gamma]$ **in a uniform way**.

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Theorem 4 [C.,Hoyrup, 2020]

The following are equivalent:

- X is countably-based;
- $[\boldsymbol{D_2}] = \boldsymbol{D_2}$ in a uniform way.

Countably-based spaces

$$[\boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}] = \boldsymbol{\Sigma}^0_{\boldsymbol{\alpha}}$$

$$[\boldsymbol{\Delta}^0_2] = \boldsymbol{\Delta}^0_2$$

$$[\boldsymbol{D_\alpha}] = \boldsymbol{D_\alpha}$$

$$[\boldsymbol{D_2}] = \boldsymbol{D_2}$$

$$[\boldsymbol{\Sigma}^0_1] = \boldsymbol{\Sigma}^0_1$$

On countably-based spaces

Case 2: real polynomials

Space of real polynomials

Consider $\mathbb{R}[X] = \bigcup_{n\in\mathbb{N}} \mathbb{R}_n[X]$ equipped with the coPolish topology:

$$O \subseteq \mathbb{R}[X] \text{ is open if:} \qquad \forall n, O \text{ is open in } \mathbb{R}_n[X]$$

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(Admissible) representation of a polynomial $P \in \mathbb{R}[X]$:

- Some bound on the degree $n \geq \deg(P)$;
- The coefficients $(p_0, ..., p_n)$ such that $P = p_0 + p_1X + ... + p_nX^n$;

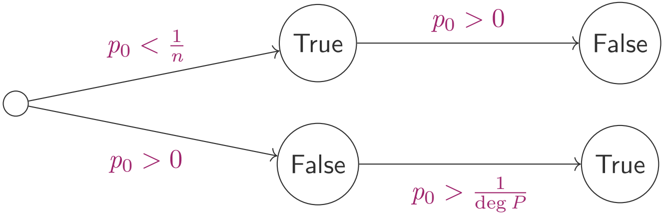
Polynomials: complexity of A

$$A = \left\{ P \in \mathbb{R}[X] : p_0 = 0 \text{ or } p_0 > \frac{1}{\deg(P)} \right\}$$

Lemma 5 Complexity of A (Symbolic)

$$A \in [\boldsymbol{D_2}]$$

Proof:



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Lemma 6 Complexity of A (Topological)

$$A \notin \boldsymbol{D_2}$$

Proof:

$$\frac{1}{m} + \frac{X^{m+1}}{p} \, (\in A) \xrightarrow[p \to +\infty]{} \frac{1}{m} \, (\notin A) \xrightarrow[m \to +\infty]{} 0 \, (\in A)$$

Complexity of B

$$B = \left\{ \frac{1}{k_1} + \frac{X^{k_1}}{k_2} + \frac{X^{k_2}}{k_3} + \cdots + \frac{X^{k_{n-2}}}{k_{n-1}} + \frac{X^{k_{n-1}}}{k_n} \; : \; k_1 < k_2 < \cdots < k_n \text{ and } n \text{ even} \right\}$$

Lemma 7 Complexity of B

- $B \in [\boldsymbol{D_2}]$;
- $B \in \boldsymbol{\Delta^0_2}$ and not below.

Complexity on the space of real polynomials

$$
\begin{array}{ccc}
[\boldsymbol{\Sigma}^0_2] & = & \boldsymbol{\Sigma}^0_2 \\
[\boldsymbol{\Delta}^0_2] & = & \boldsymbol{\Delta}^0_2 \\
\vdots & \nearrow & \vdots \\
[\boldsymbol{D}_2] & \hookleftarrow & \boldsymbol{D}_2 \\
[\boldsymbol{\Sigma}^0_1] & = & \boldsymbol{\Sigma}^0_1
\end{array}
$$

On $\mathbb{R}[X]$

Topology vs. sequentiality

Why is there a difference between topological and symbolic complexity?

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Partial explanation:

The difference is related to the mismatch between **topological** and **sequential** aspects of the space.

continuity	sequential continuity
compactness	sequential compactness
closure	sequential closure

Topology vs. sequentiality

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PARTIAL EXPLANATION:

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Indeed:

Theorem 8 [C.,Hoyrup, 2020]

For X a coPolish space, the following are equivalent:

- Closure and sequential closure coincide on X (Fréchet-Urysohn);
- For every $n \in \mathbb{N}$, $[\boldsymbol{D_n}] = \boldsymbol{D_n}$.

Case 3: spaces of open sets

Spaces of open sets

The category of admissibly represented spaces is cartesian closed. In particular, if X is admissibly represented, then $\mathcal{O}(X)$ (equipped with Scott topology) is too [Schröder, 2015].

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Theorem 9 [Hoyrup, 2020]

For X a represented space, on $\mathcal{O}(X)$,

$$\forall n \in \mathbb{N}, \quad \boldsymbol{D_n} = [\boldsymbol{D_n}]$$

(Non-effective proof)

$$[\boldsymbol{D_\omega}] \hookleftarrow \boldsymbol{D_\omega}$$

...

$$[\boldsymbol{D_2}] = \boldsymbol{D_2}$$

$$[\boldsymbol{\Sigma^0_1}] = \boldsymbol{\Sigma^0_1}$$

Open sets of Polish spaces

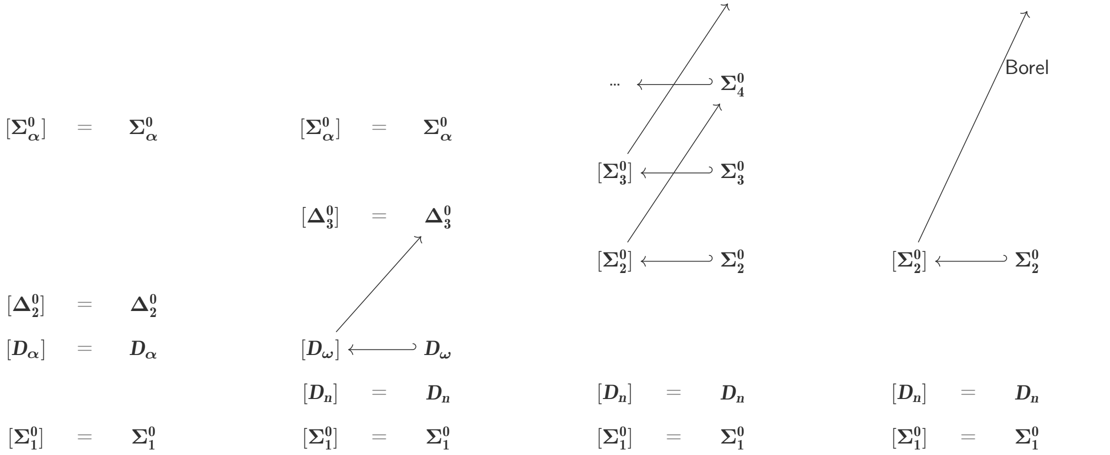
What about higher complexities?

Open sets of Polish spaces

What about higher complexities? [Hoyrup, 2020]

Let $X_{nk} = \{x \in X : x$ has no compact neighborhood$\}$.

Class I : $X_{nk} = \emptyset$ | **Class II: X_{nk} finite** | **Class III: σ-compact** | **Class IV: not σ-compact**



Partial conclusion

Countably-based spaces	▶ $[\Gamma] = \Gamma$;
$\mathbb{R}[X]$	▶ there exists some $A \in [\boldsymbol{D_2}]$ with A not below $\boldsymbol{\Delta^0_2}$; ▶ for every α, $[\boldsymbol{\Sigma^0_\alpha}] = \boldsymbol{\Sigma^0_\alpha}$;
$\mathcal{O}(X)$	▶ $[\boldsymbol{D_n}] = \boldsymbol{D_n}$ for $n \in \mathbb{N}$: well-behaved low complexity; ▶ In some cases, $[\boldsymbol{\Sigma^0_\alpha}]$ and $\boldsymbol{\Sigma^0_\alpha}$ disagree at low levels, then agree; ▶ In some others, they never agree.

Partial conclusion

Countably-based spaces

- $[\Gamma] = \Gamma$;

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- there exists some $A \in [\boldsymbol{D_2}]$ with A not below $\boldsymbol{\Delta^0_2}$;
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- $[\boldsymbol{D_n}] = \boldsymbol{D_n}$ for $n \in \mathbb{N}$: well-behaved low complexity;
- In some cases, $[\boldsymbol{\Sigma^0_\alpha}]$ and $\boldsymbol{\Sigma^0_\alpha}$ disagree at low levels, then agree;
- In some others, they never agree.

PARTIAL EXPLANATION (AGAIN):

The difference is related to the mismatch between **topological** and **sequential** aspects of the space.

Hardness

Hardness in Polish spaces

How can we show that a set A is **not** in Σ_2^0?
$\rightarrow$ **Reductions**: you prove that A is "harder than" any $\mathbf{\Pi_2^0} = \check{\mathbf{\Sigma}}_\mathbf{2}^\mathbf{0}$ set of $\mathbb{N}^{\mathbb{N}}$.

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Definition 10 Wadge reducibility

- For $A \subseteq X$ and $B \subseteq Y$, A is **Wadge reducible** to B (written $A \leq_W B$) if:

$$\exists f : X \mapsto Y, \quad x \in A \iff f(x) \in B$$

- $A \subseteq X$ is Γ-hard if:

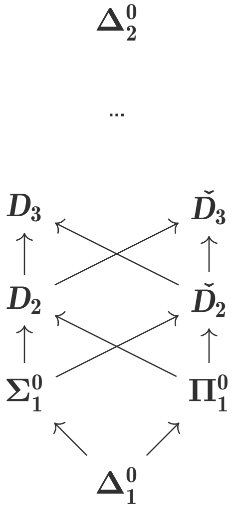
$$\forall C \in \Gamma\left(\mathbb{N}^\mathbb{N}\right), C \leq_W A$$

Hardness in Polish spaces

Lemma 11 (Wadge Lemma)

For any Borel subset A of a Polish space,

$$A \notin \Gamma \iff A \text{ is } \check{\Gamma}\text{-hard}$$



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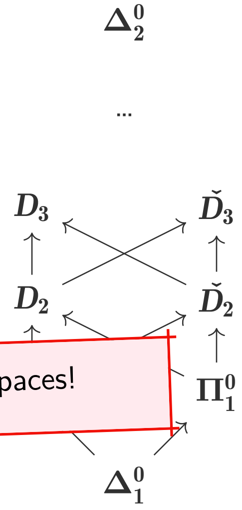
Not true (in general) outside of Polish spaces!



Hardness on represented spaces

Hardness captures symbolic complexity:

Theorem 12

For X a represented space, and $A \subseteq X$ Borel,

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To capture topological complexity, weakened version of hardness:

Definition 13 [Hoyrup, 2020]

$A \subseteq X$ is Γ-**hard*** if

for every countably-based weaker topology τ,

A is Γ-hard in (X, τ).

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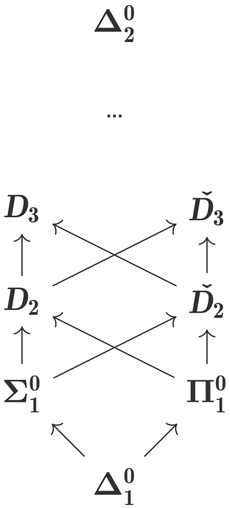

Conclusion

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Answer:

- $\Gamma = [\Gamma]$ on countably-based spaces;
- They differ in general.

Partial explanation:

The difference is related to the mismatch between **topological** and **sequential** aspects of the topology.

- Weaker notion of hardness to capture topological instead of symbolic complexity.

Thank you

Questions?