The aperiodic Domino problem in (almost) every dimension

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Subshifts

Definition 1 Subshifts

A $\mathbb{Z}^d$ *subshift* is a set of colorings $\mathbb{Z}^d \mapsto \Sigma$ defined by forbidden patterns $\mathcal{F}$:

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

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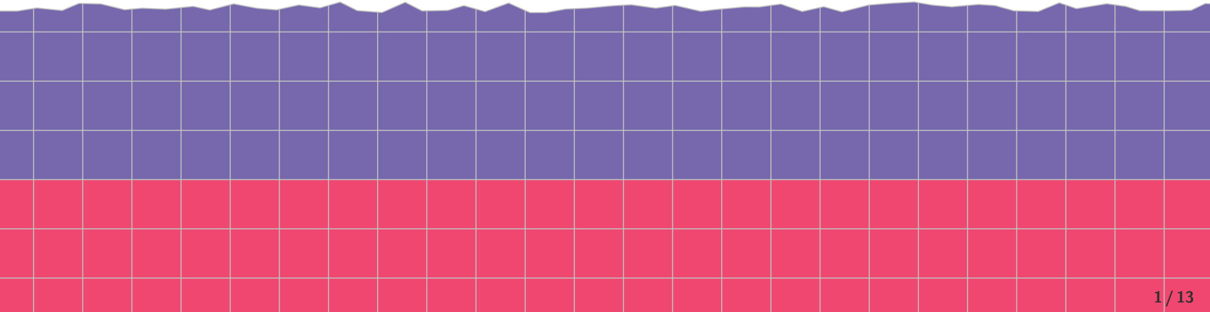

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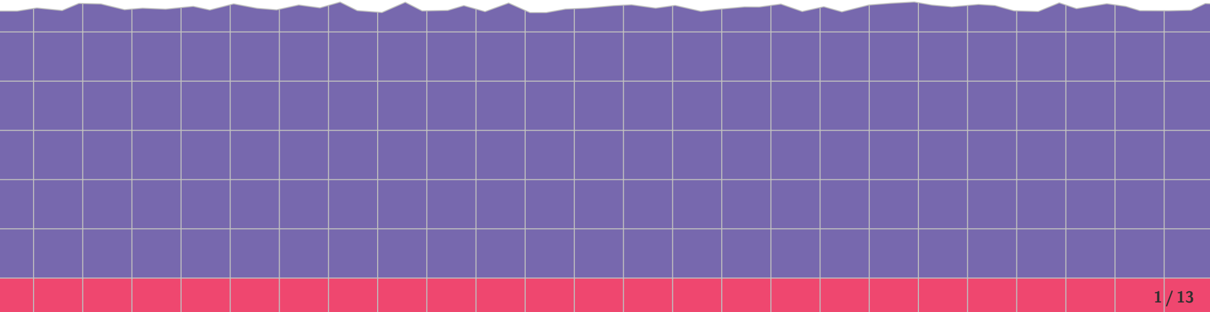

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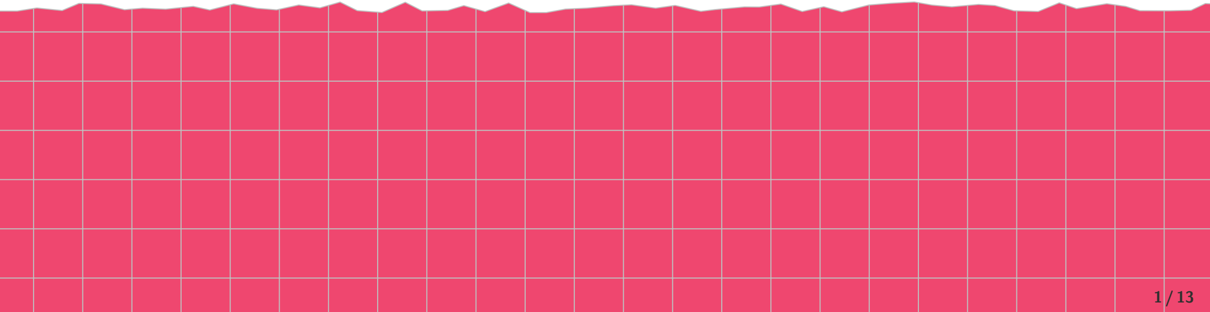

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We consider three classes of subshifts:

- Subshifts of Finite Type (SFTs): $X_{\mathcal{F}}$ for some finite $\mathcal{F}$.
- Effective subshifts: $X_{\mathcal{F}}$ for some recursively enumerable $\mathcal{F}$.
- Sofic subshifts: X such that there exists $X' \subseteq (\Sigma')^{\mathbb{Z}^d}$, $\pi : \Sigma' \mapsto \Sigma$, $X = \pi(X')$.

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SFT $\implies$ sofic $\implies$ effective $\implies$ $(d+1)$-sofic [Hoch-2010,AS-2013,DRS-2012].

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Input A SFT (= symbols + finite set of forbidden pattern)
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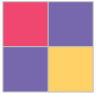
The Domino problem is co-recursively enumerable (ie. Π_1^0).



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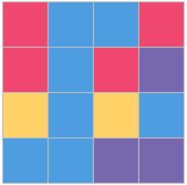

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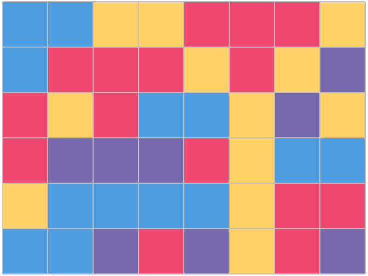

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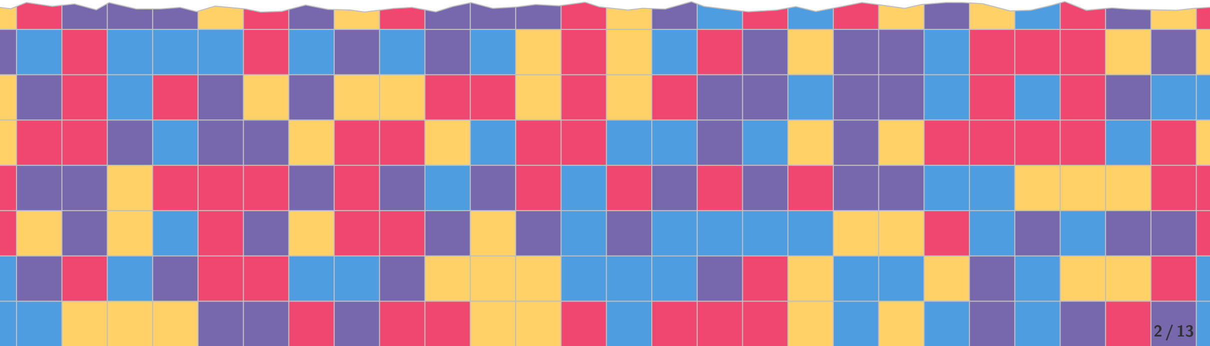

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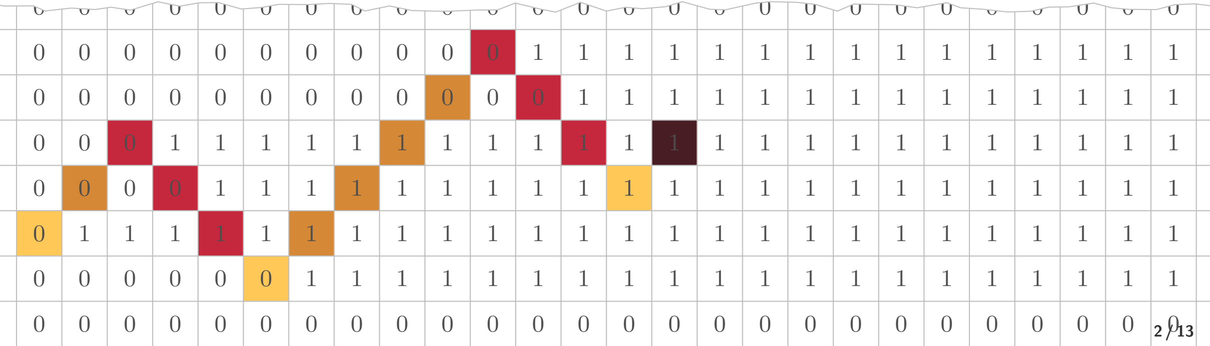

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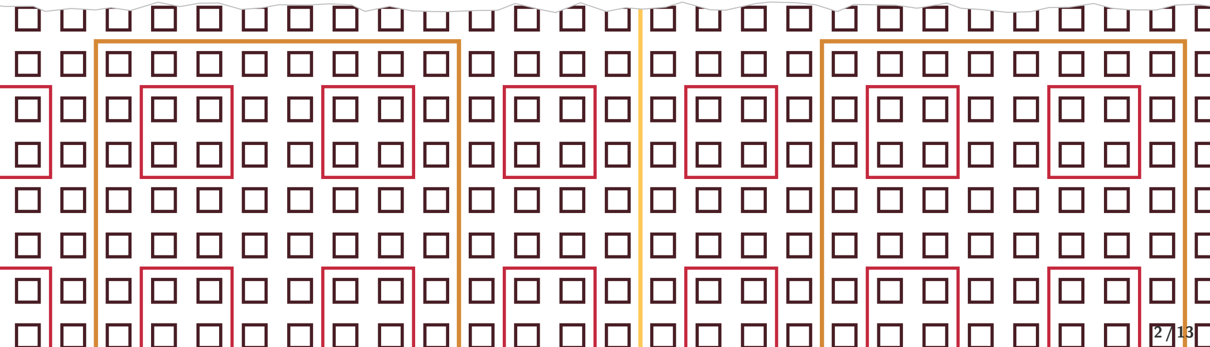

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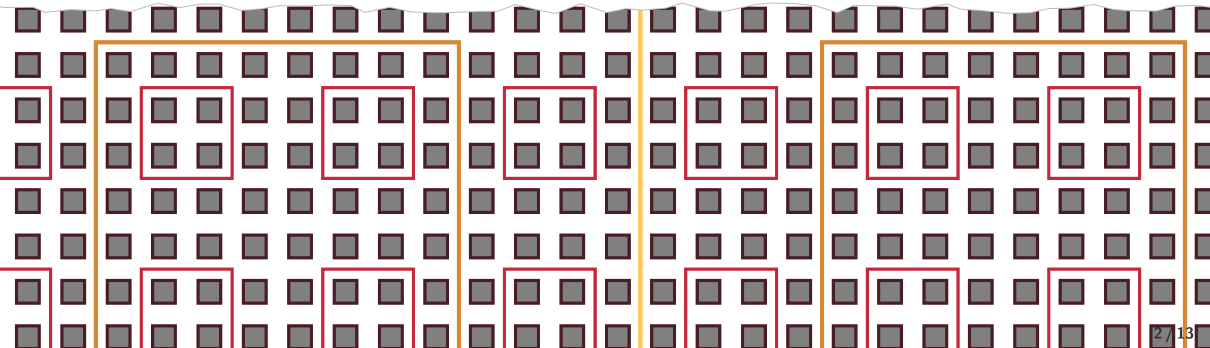

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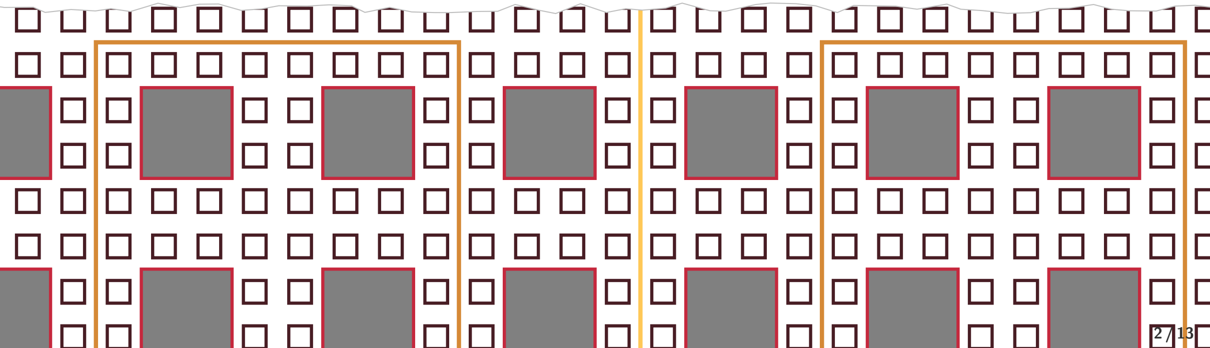

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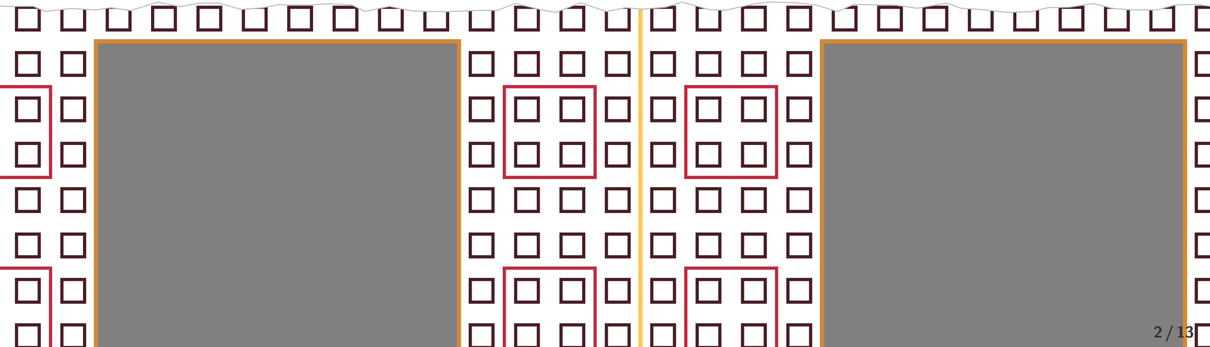

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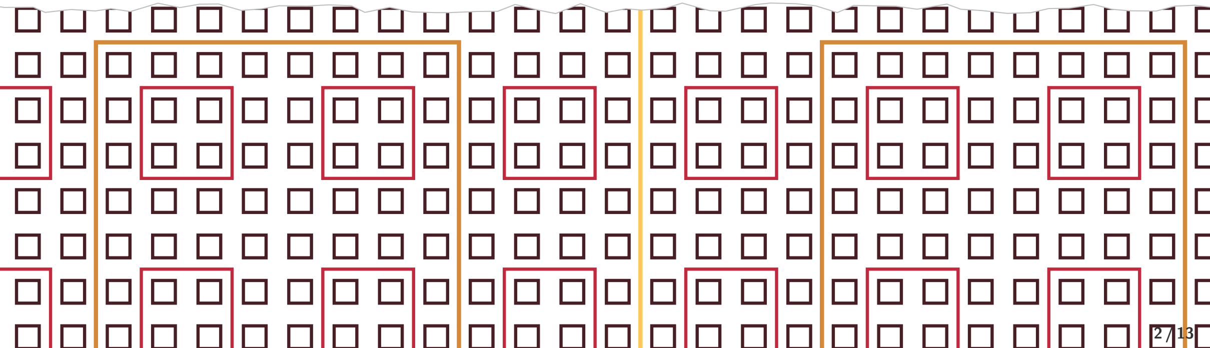

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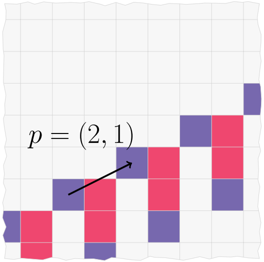
Aperiodic Domino problem

Definition 3 (A)periodicity

1. A coloring $x \in \Sigma^{\mathbb{Z}^d}$ is *periodic* of period $p \in \mathbb{Z}^d$ if: $\forall i \in \mathbb{Z}^d, x_{i+p} = x_i$.



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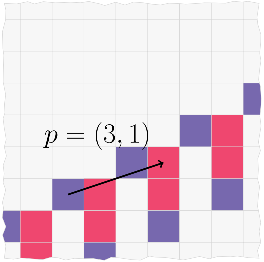

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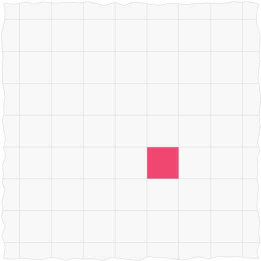

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Aperiodic Domino problem:

Input An effective $\mathbb{Z}^d$ subshift.

Output Is there an admissible *aperiodic* coloring?

What can we say about the aperiodic Domino?

Aperiodic Domino problem on $\mathbb{Z}^2$ [GHV18]

[GHV18] On $\mathbb{Z}^2$: The shepherd of periods

Consider a $\mathbb{Z}^2$ coloring that is neither $p_0 = (2,2)$ nor $p_1 = (2,0)$ periodic. Then:

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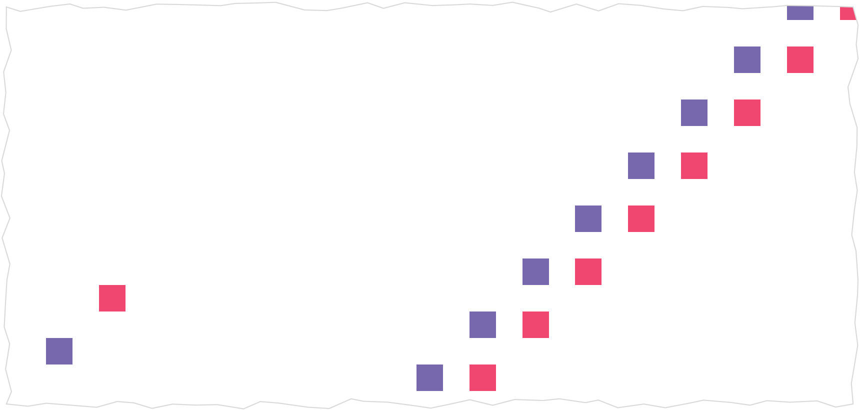

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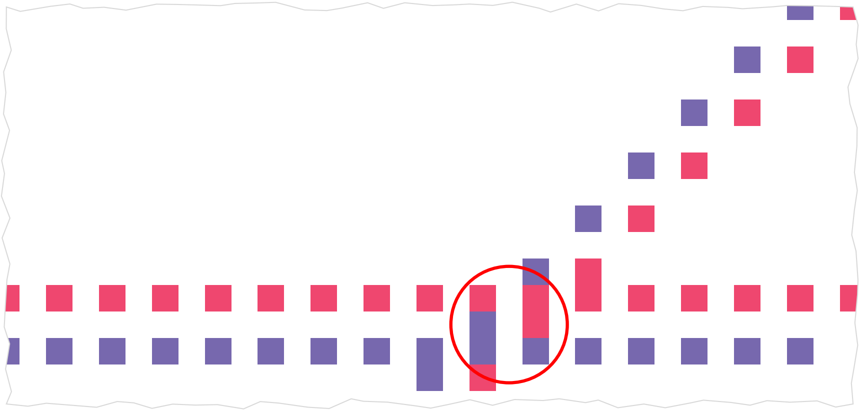

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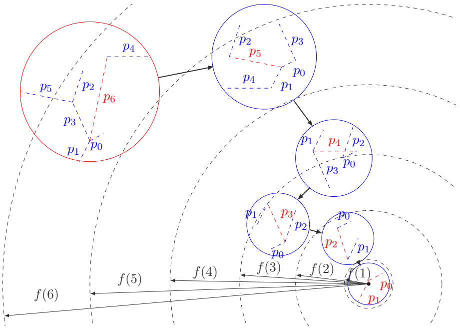

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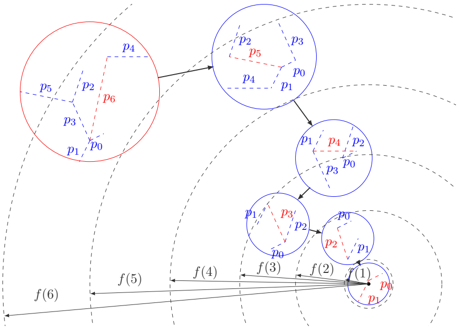
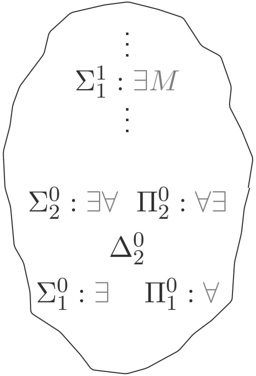


Corrolary 4 [GHV18,Corollary 8]

Aperiodic Domino is co-recursively enumerable (ie. Π_1^0) for $\mathbb{Z}^2$ subshifts.

Aperiodic Domino problem on $\mathbb{Z}^d, d \gtrsim 3$

Aperiodic Domino in higher dimension



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Theorem 5 AD in high dimension is not arithmetic

Aperiodic Domino is Σ_1^1-complete for:

- $\mathbb{Z}^d$ sofic subshifts with $d \geq 3$.
- $\mathbb{Z}^d$ SFTs with $d \geq 4$.

We reduce **State recurrence**:

Input A non-deterministic TM and a state q_0

Output Is there a run on ε that visits q_0 infinitely often?

On $\mathbb{Z}^3$: From AD to state recurrence (Part 1)

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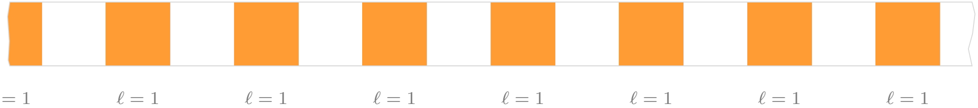

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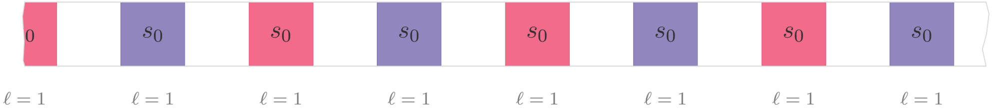

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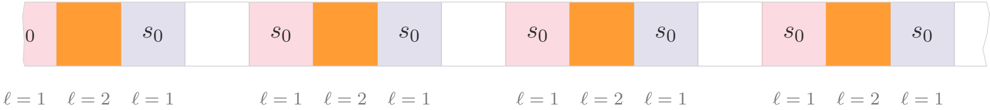

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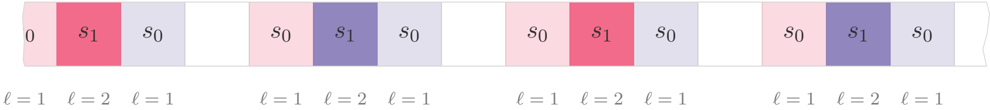

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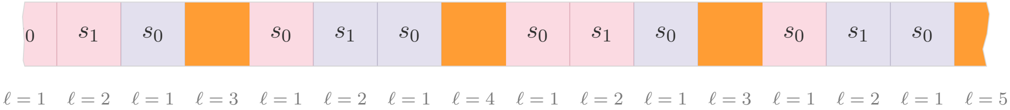

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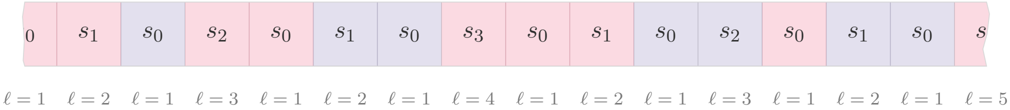

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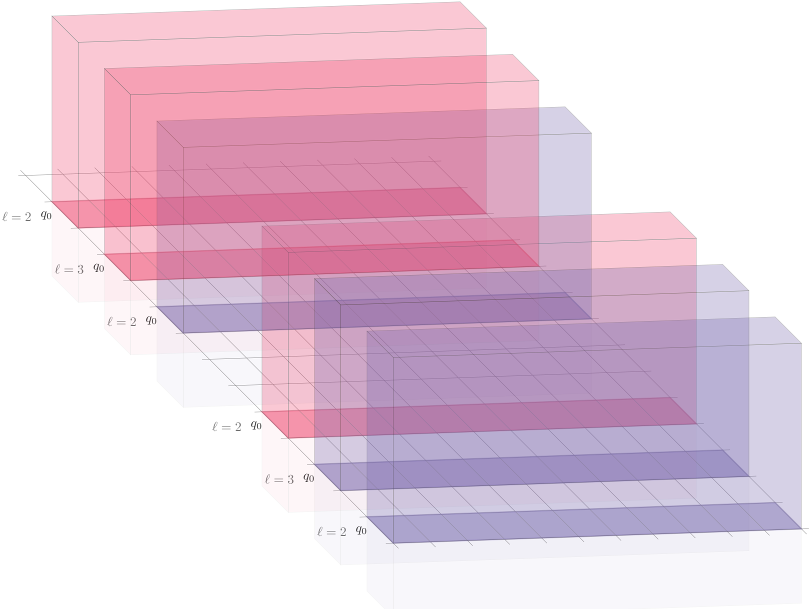

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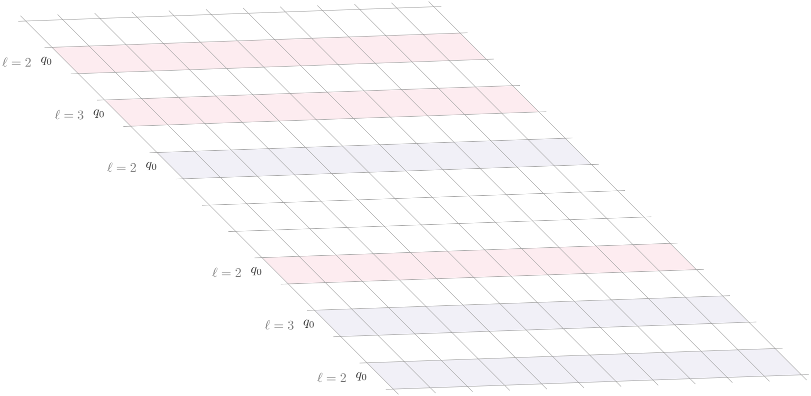

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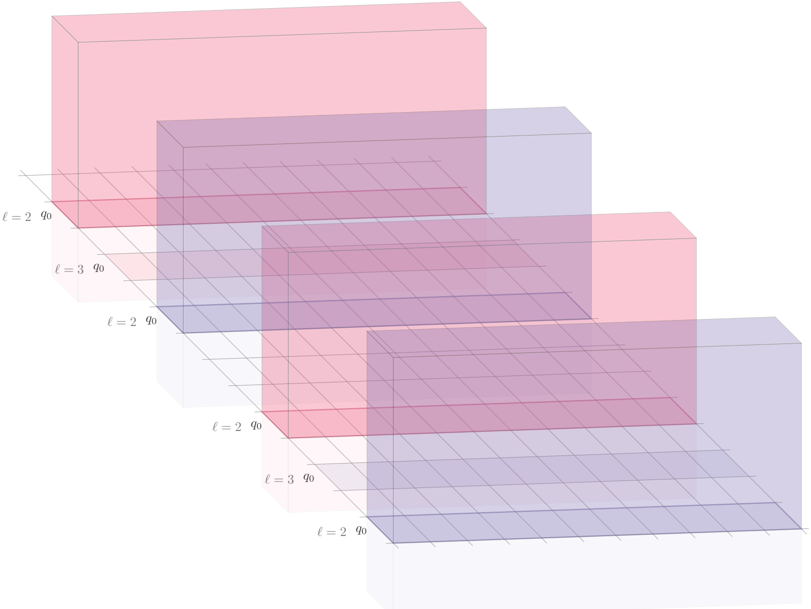

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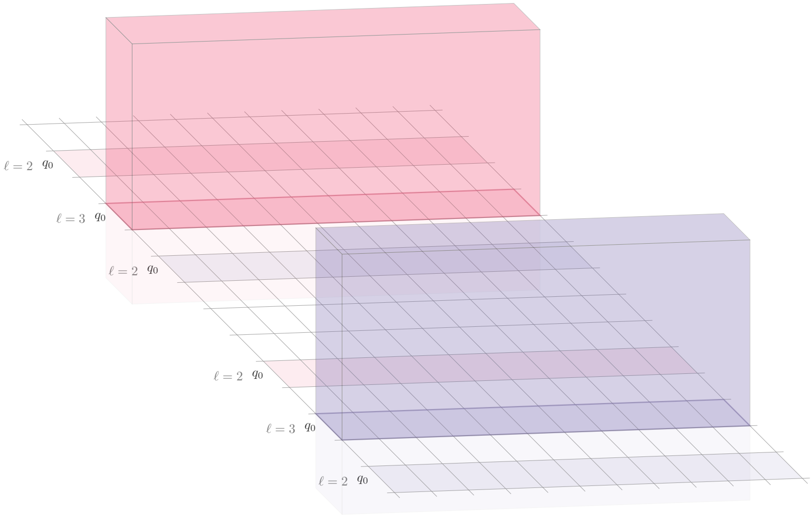

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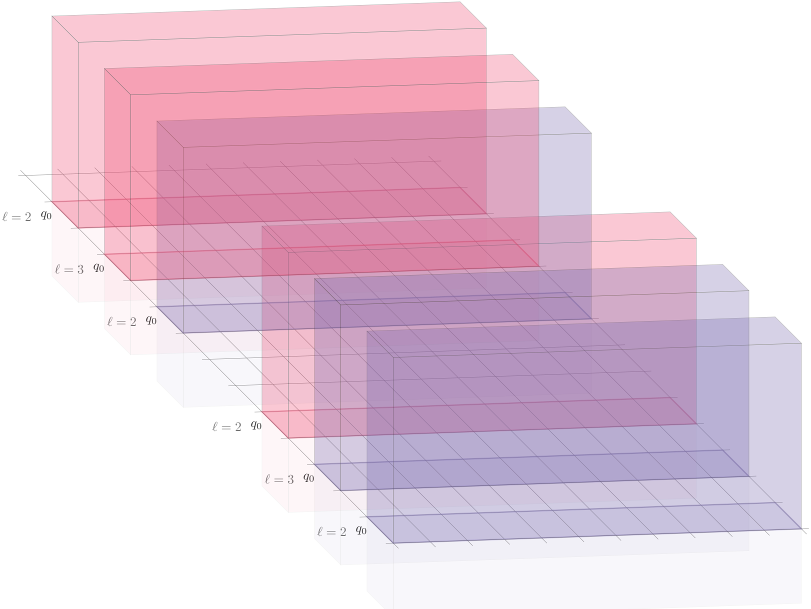

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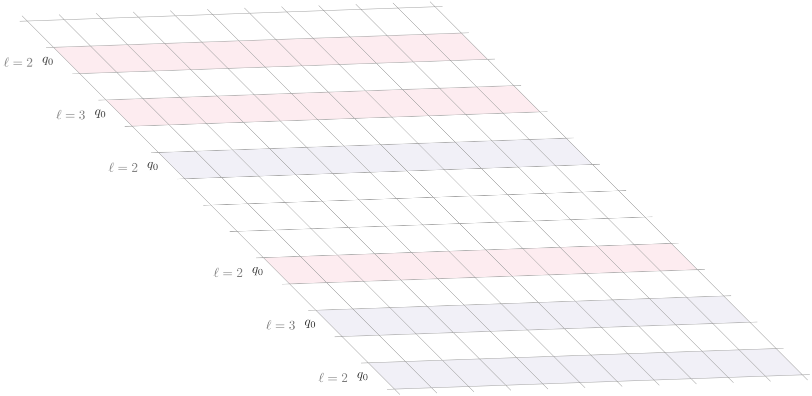

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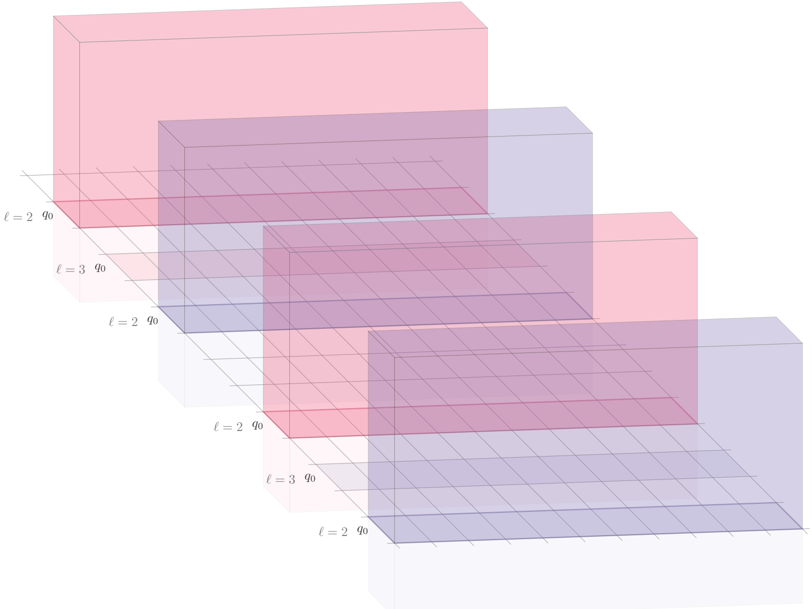

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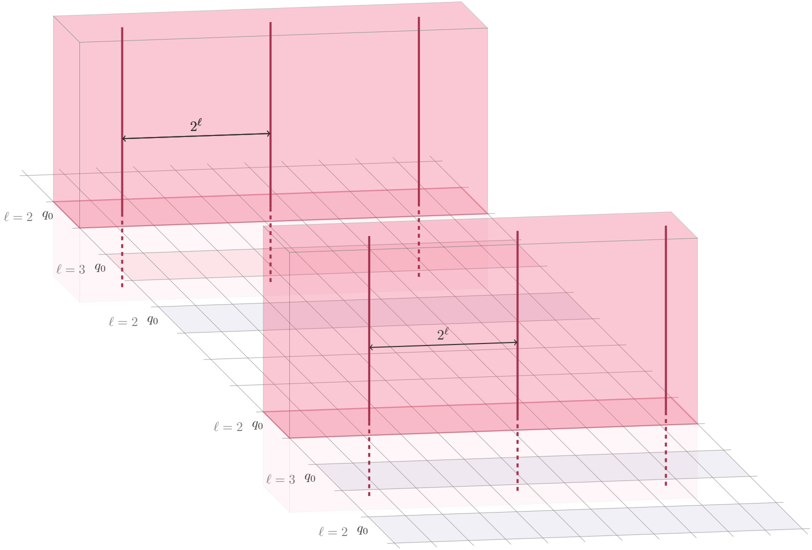

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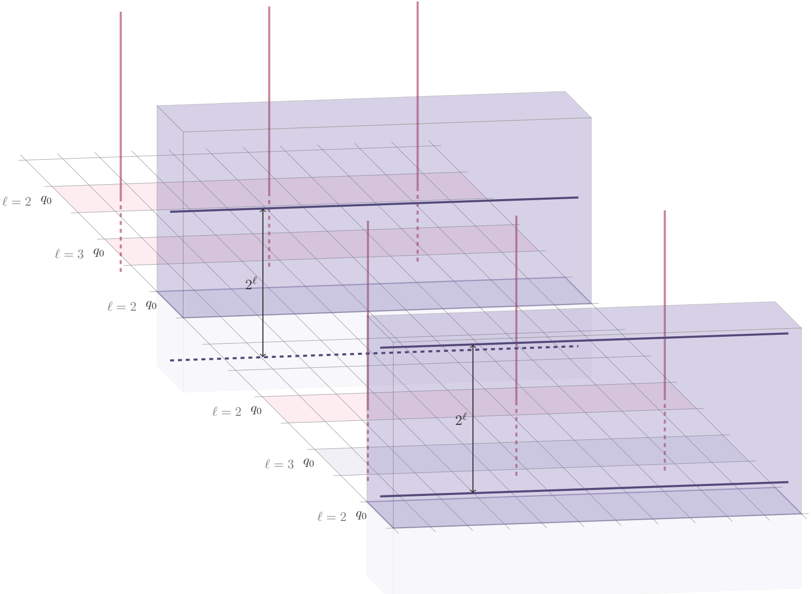

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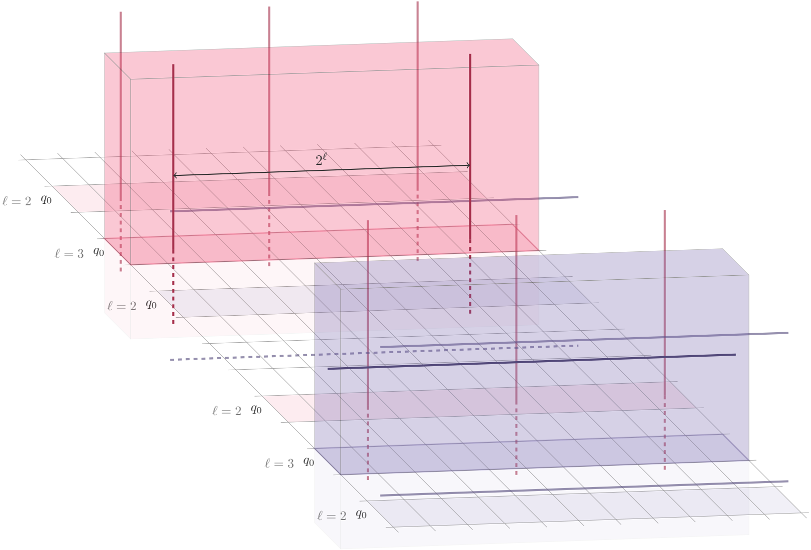

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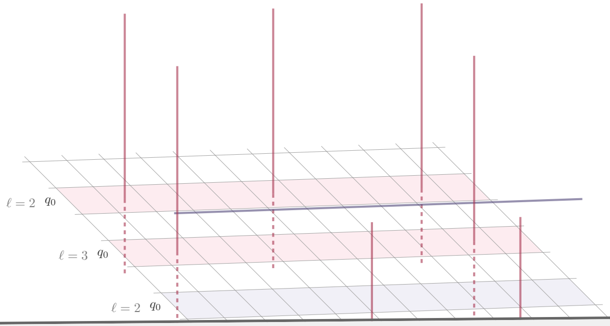

Aperiodic coloring $\iff$ the machine visits q_0 infinitely often.

Why two additional dimensions? (Spoiler: I lied)

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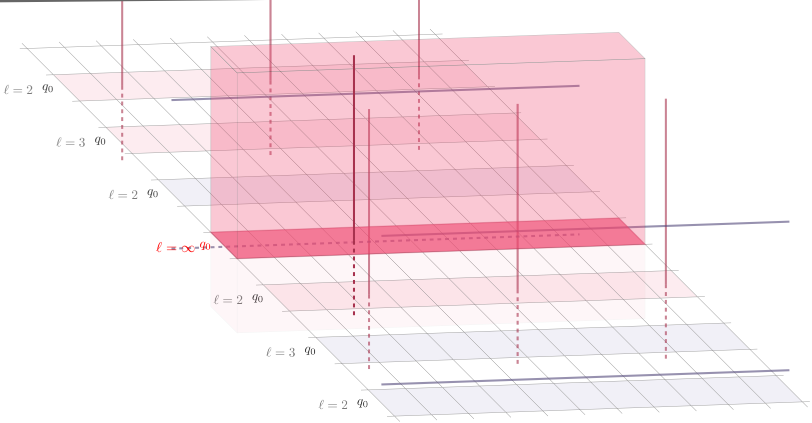
Beware the slice of ∞ level!



Recap and consequences

Aperiodic Domino problem:

Input An effective $\mathbb{Z}^d$ subshift.

Output Is there an admissible *aperiodic* coloring?

Its (computational) complexity depends on the dimension of the subshift.

$\Longrightarrow$: separates $2, 3$ and 4-dimensional subshifts.

Dimension / type	2D	3D	4D+
finite type	Π_1^0	**open**	Σ_1^1
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Difficulty of the Domino problem

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Difficulty of the Domino problem

Aperiodic Domino problem on $\mathbb{Z}^3$ SFTs

Two lines must cross in $\mathbb{Z}^3$ SFTs

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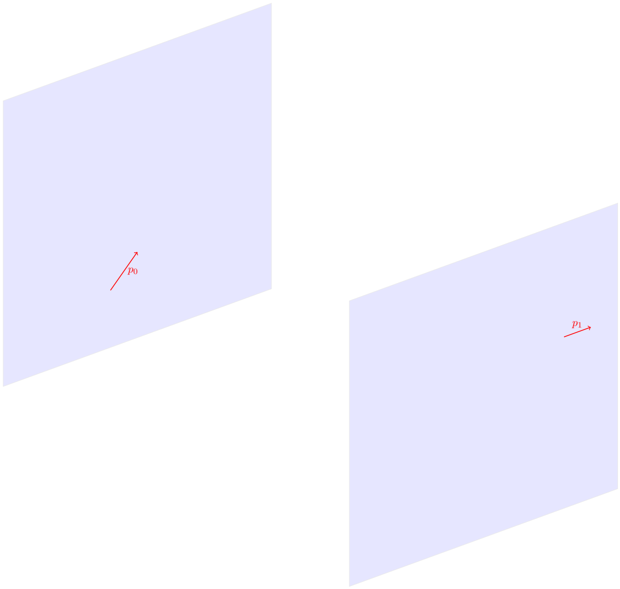

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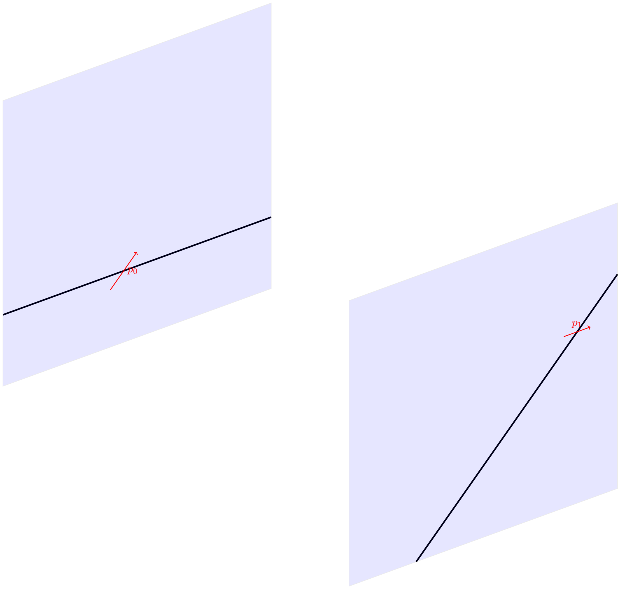

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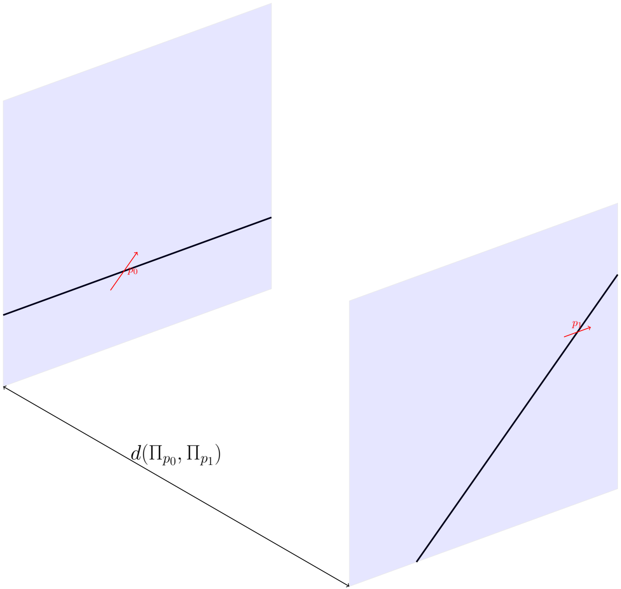

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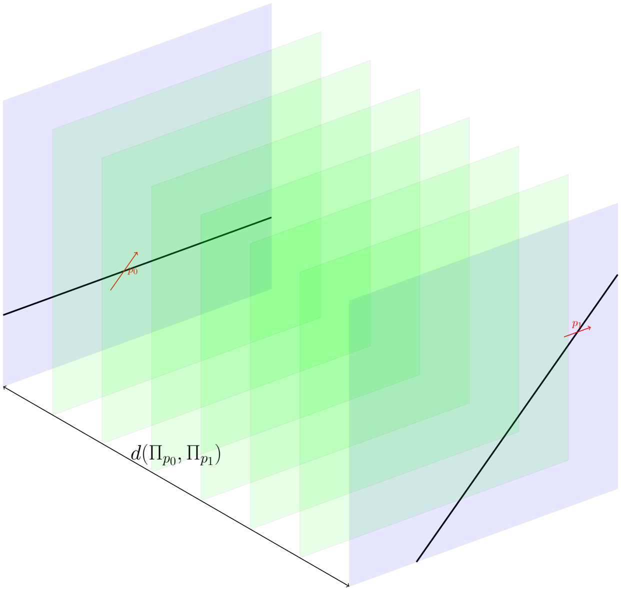

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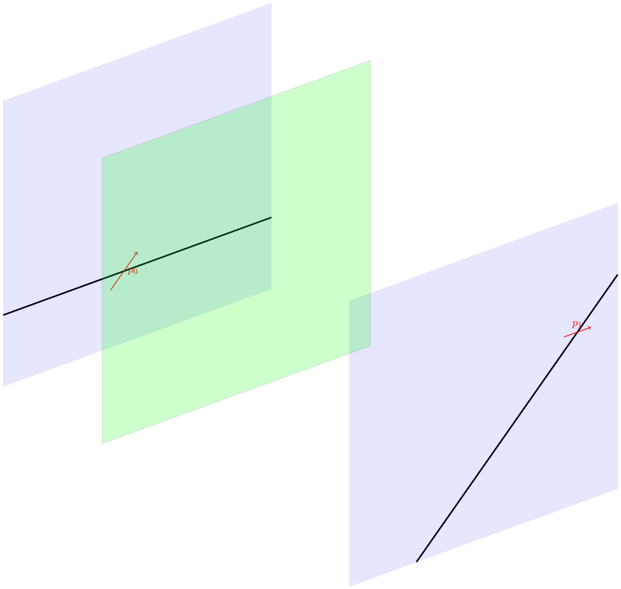

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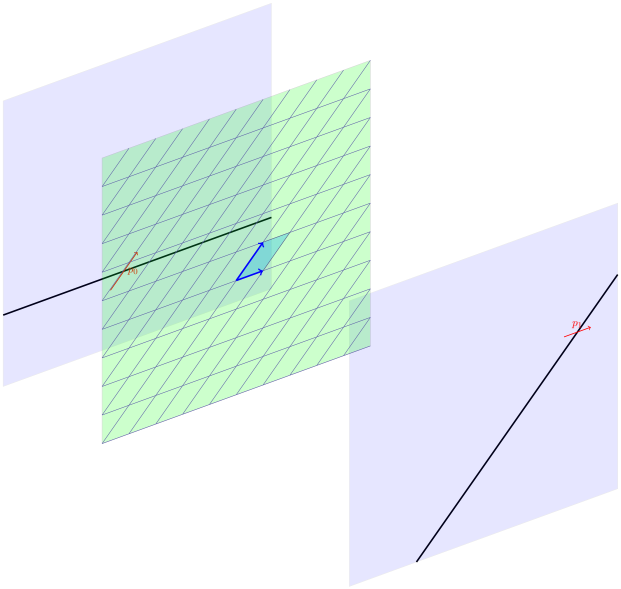

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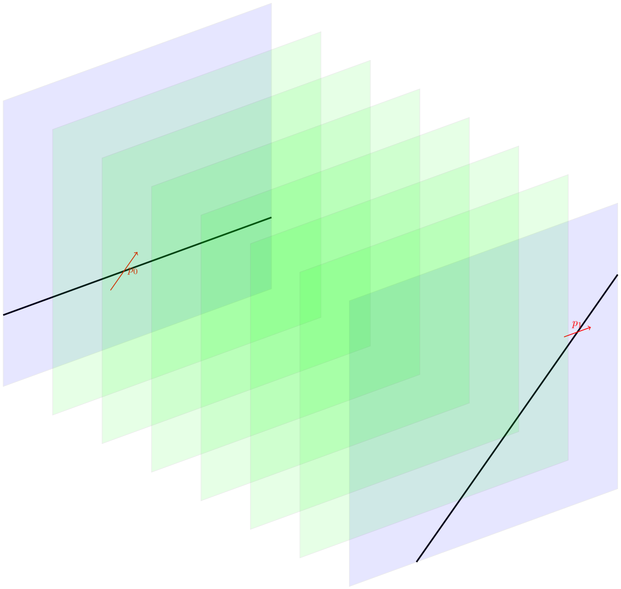

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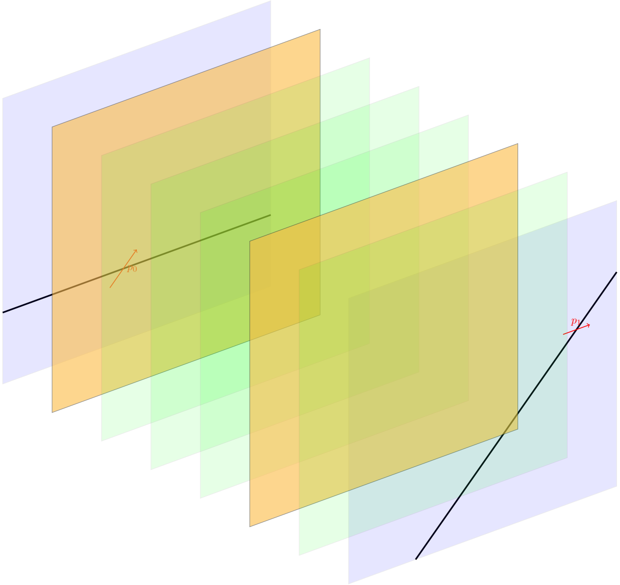

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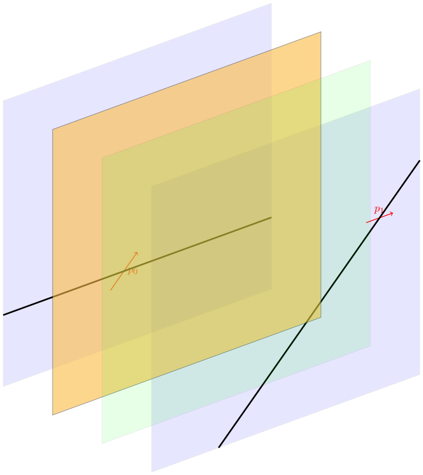

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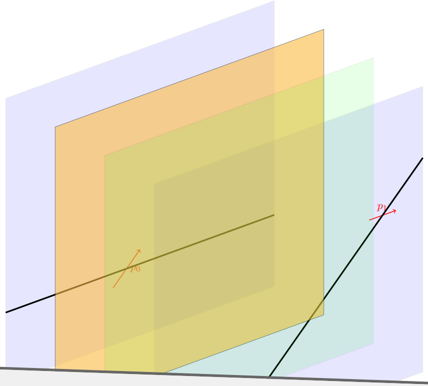

Breaks in balls of size

$$\simeq |\Sigma|^{\|p_0 \wedge p_1\|}$$

AD on $\mathbb{Z}^3$ SFTs

Do $\mathbb{Z}^3$ SFTs behave similarly to $\mathbb{Z}^2$ subshifts in terms of aperiodicity? i.e.

Conjecture 6

There exists a computable function f such that any $\mathbb{Z}^3$ SFT $X \in$ **EAC** contains **a configuration which breaks any period** $\|p\| \leq n$ inside the centered square of edge $f(W, |\Sigma|, n)$.

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Why?

- Two lines of period breaks do not cross on $\mathbb{Z}^3$ in general... But they do in $\mathbb{Z}^3$ SFTs.
- Embedding computations in SFTs requires two dimensions.
- By compactness (i.e. limits), the two additional dimensions of the Σ_1^1-proof *are necessary*.

Aperiodic Domino problem and entropies

Complexity function

Definition 7 Complexity function

The *complexity function* $N_X(n)$ is defined as the number of different patterns of size n^d that appear in $X \subseteq \Sigma^{\mathbb{Z}^d}$.

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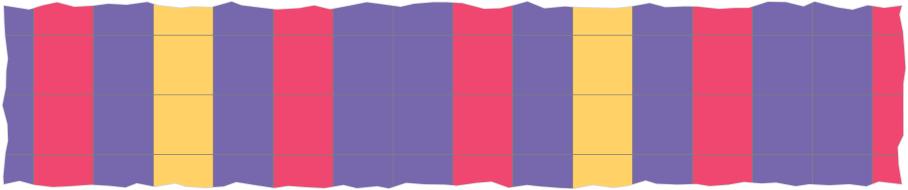
Example : the SFT X defined by



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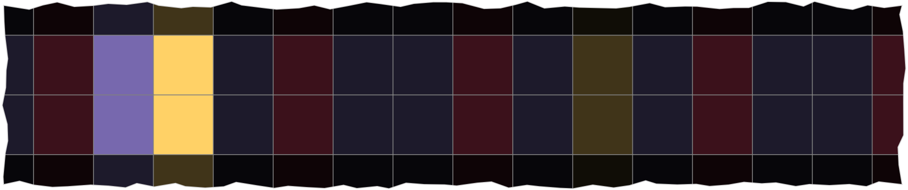

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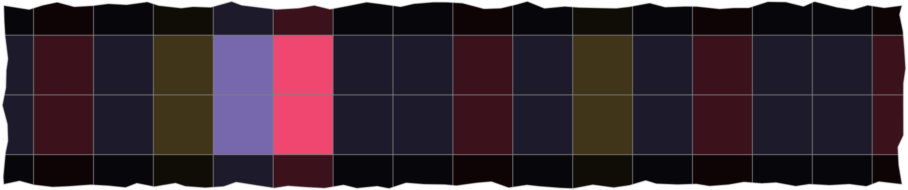

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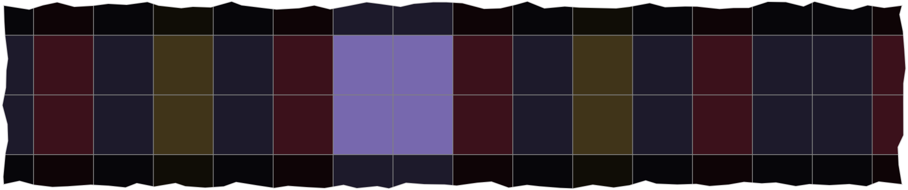

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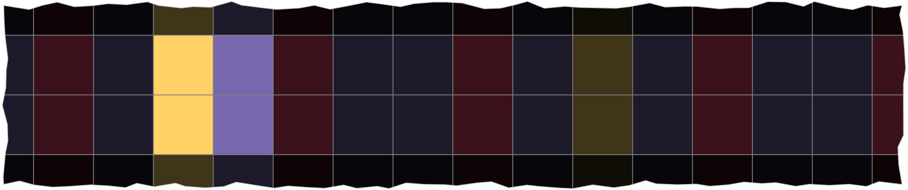

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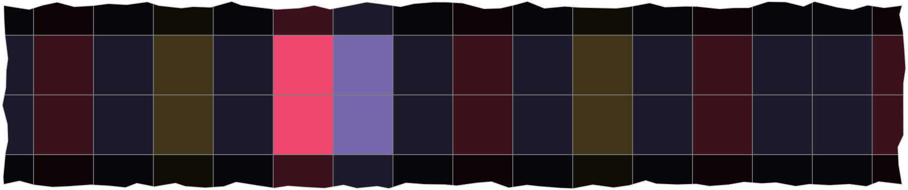

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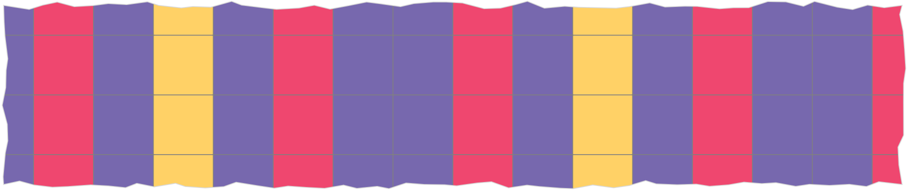

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Define the m-entropy:

$$h_m(X) = \limsup_{n \to +\infty} \frac{\log N_X(n)}{n^m}$$

Theorem 8 Improves [Hochman, 2009]

Let X be a $\mathbb{Z}^d$ subshift. If $h_{d-1}(X) = +\infty$, then there exists an aperiodic configuration in X.

Aperiodic Domino and entropies

For X a $\mathbb{Z}^d$ subshift, $h_{d-1}(X) = \limsup_{n\to+\infty} \frac{\log N_X(n)}{n^{d-1}}$ implies that X contains an aperiodic configuration.

Proof:

1. If $h_{d-1}(X) = +\infty$, for every $k \in \mathbb{N}$, there exists a border of thickness k with at least two admissible completions of its interior.

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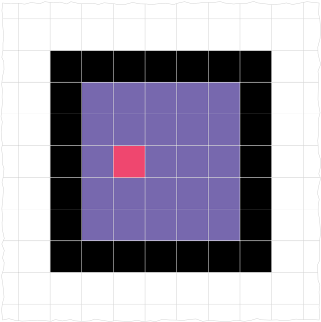

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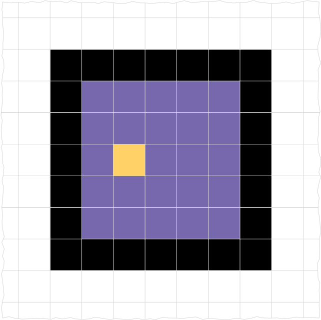

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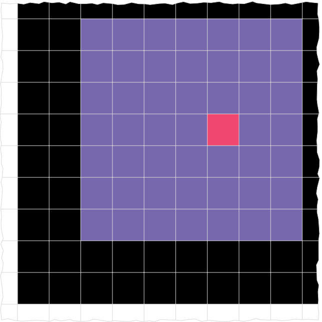

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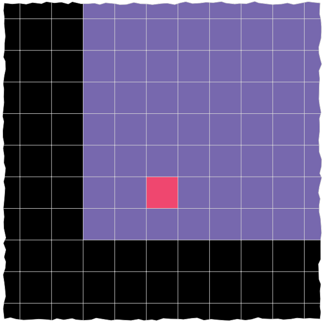

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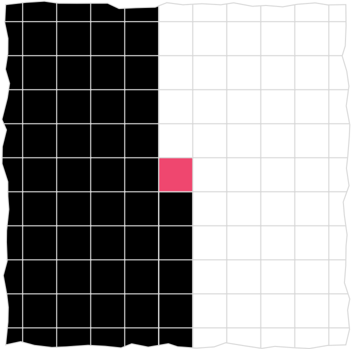

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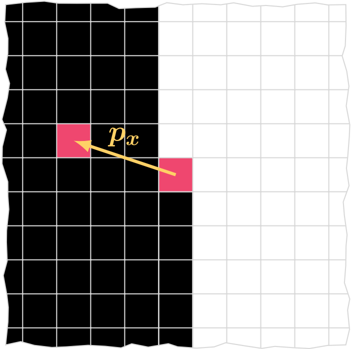

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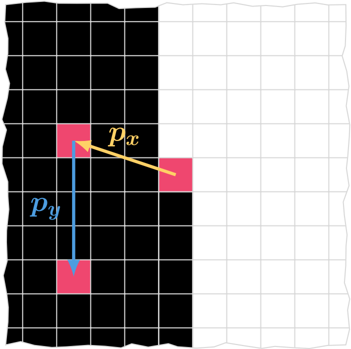

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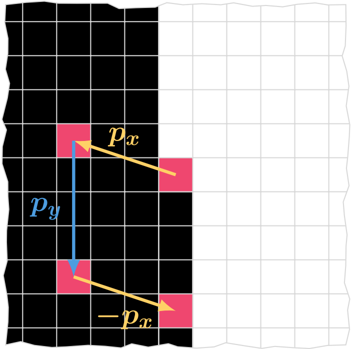

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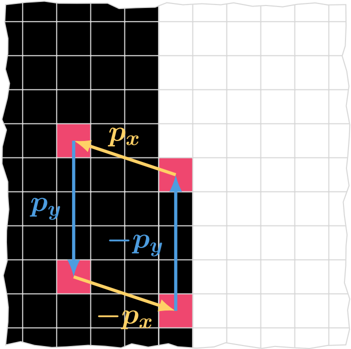

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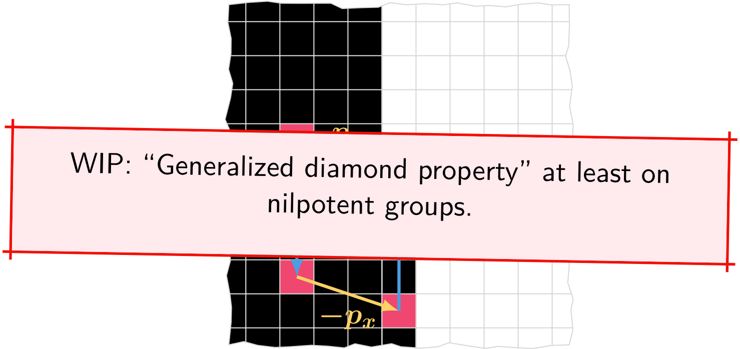

Conclusion

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Difficulty of the Domino problem

Relates to entropies:

For X a $\mathbb{Z}^d$ subshift, if $h_{d-1}(X) = +\infty$, then X contains an aperiodic configuration.

Thank you

Questions?

école normale supérieure paris–saclay