

# The aperiodic Domino problem in (almost) every dimension

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école \_\_\_\_\_  
normale \_\_\_\_\_  
supérieure \_\_\_\_\_  
paris-saclay \_\_\_\_\_

**LISN** 

# Subshifts

## Definition 1

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A  $\mathbb{Z}^d$  subshift is a set of colorings  $\mathbb{Z}^d \mapsto \Sigma$  defined by forbidden patterns  $\mathcal{F}$ :

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

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We consider three classes of subshifts:

- ▶ Subshifts of Finite Type (SFTs):  $X_{\mathcal{F}}$  for some finite  $\mathcal{F}$ .
- ▶ Effective subshifts:  $X_{\mathcal{F}}$  for some recursively enumerable  $\mathcal{F}$ .
- ▶ Sofic subshifts:  $X$  such that there exists  $X' \subseteq (\Sigma')^{\mathbb{Z}^d}$ ,  $\pi : \Sigma' \mapsto \Sigma$ ,  $X = \pi(X')$ .

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SFT  $\implies$  sofic  $\implies$  effective  $\implies$   $(d + 1)$ -sofic [Hoch-2010,AS-2013,DRS-2012].

## (Classical) Domino problem

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Computability of the Domino problem

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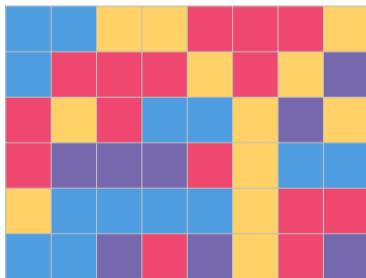
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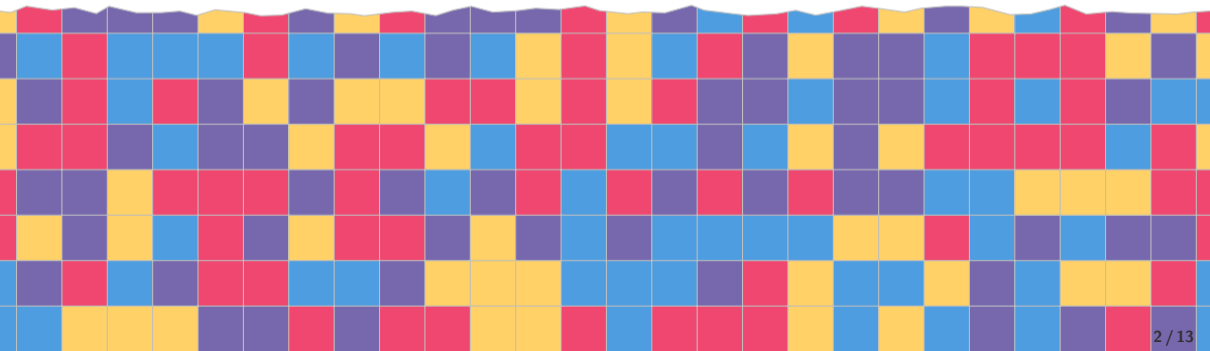
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|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
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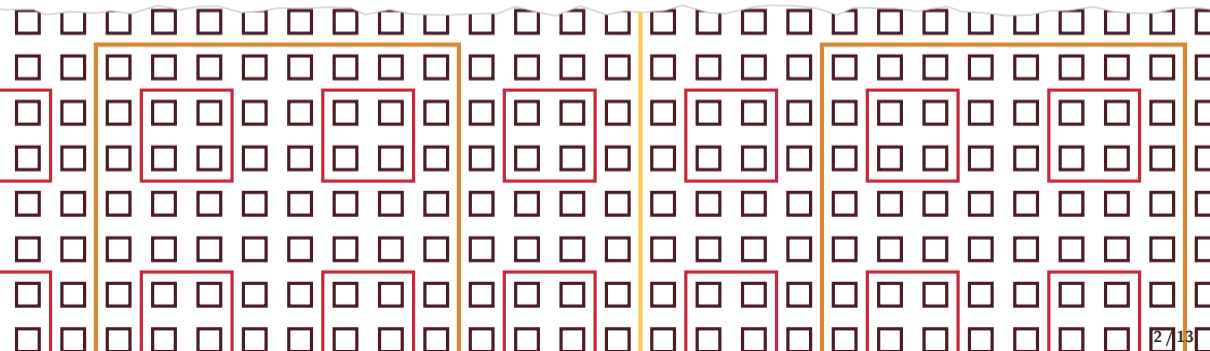
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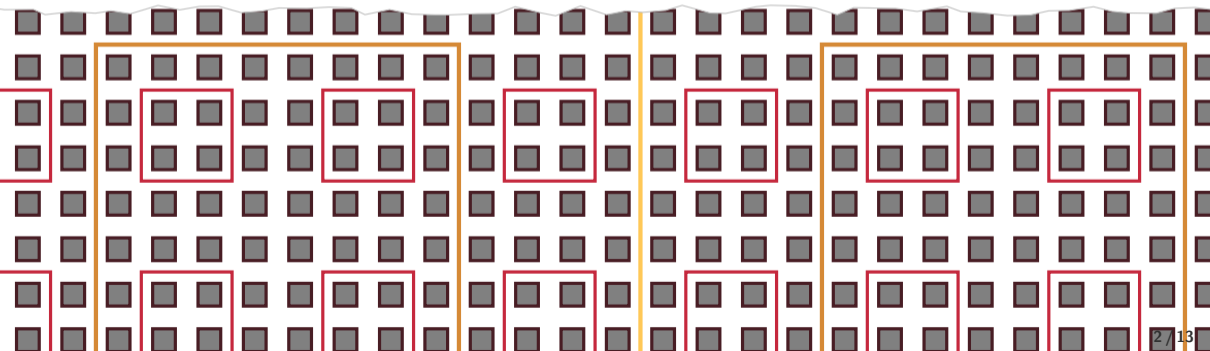
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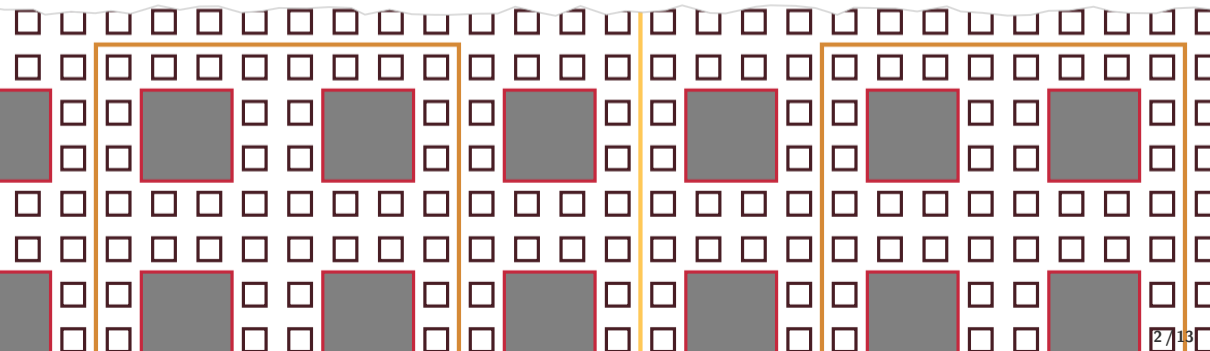
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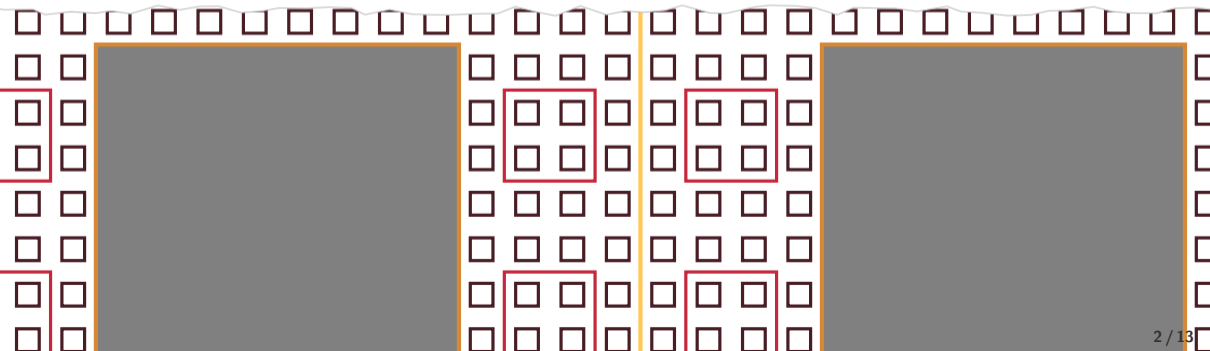
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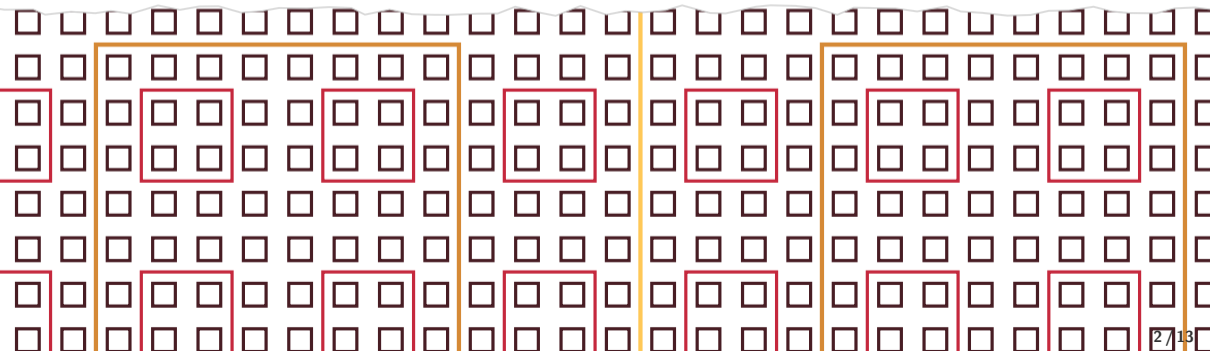
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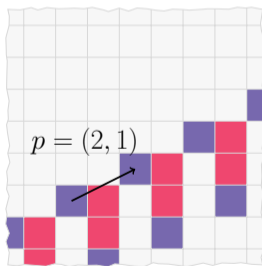
**Aperiodic colorings!**

# Aperiodic Domino problem

## Definition 3

## (A)periodicity

1. A coloring  $x \in \Sigma^{\mathbb{Z}^d}$  is *periodic* of period  $p \in \mathbb{Z}^d$  if:  $\forall i \in \mathbb{Z}^d, x_{i+p} = x_i$ .

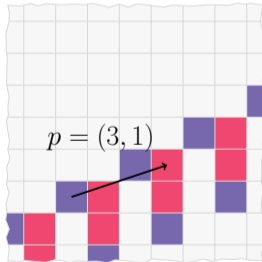


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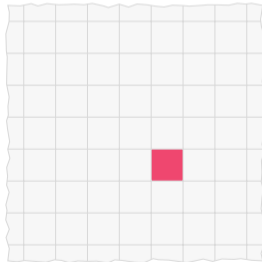


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## Aperiodic Domino problem:

**Input** An effective  $\mathbb{Z}^d$  subshift.

**Output** Is there an admissible *aperiodic* coloring?

What can we say about the aperiodic Domino?

The background of the slide features a grid of blue squares. Overlaid on this grid are orange paths that connect the corners of the squares. These paths form a complex, non-repeating pattern, which is a visual representation of the aperiodic domino problem. The paths are composed of horizontal and vertical segments, with some segments extending across multiple squares. The overall effect is a dense, interconnected network of lines that does not exhibit any simple periodicity.

**Aperiodic Domino problem on  $\mathbb{Z}^2$  [GHV18]**

## [GHV18] On $\mathbb{Z}^2$ : The shepherd of periods

Consider a  $\mathbb{Z}^2$  coloring that is neither  $p_0 = (2, 2)$  nor  $p_1 = (2, 0)$  periodic. Then:





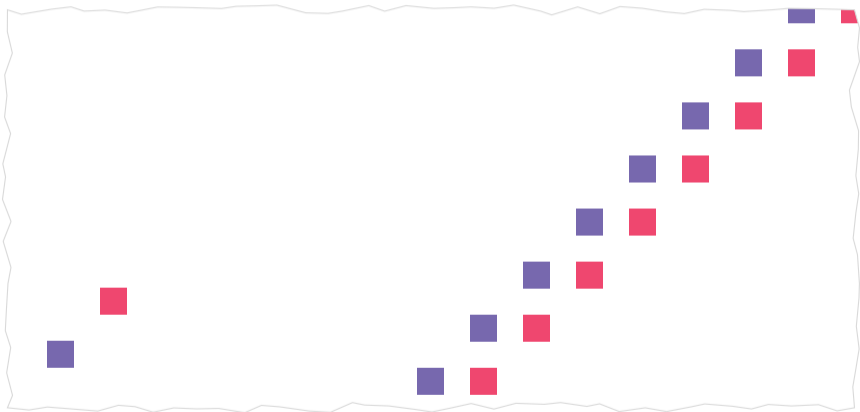
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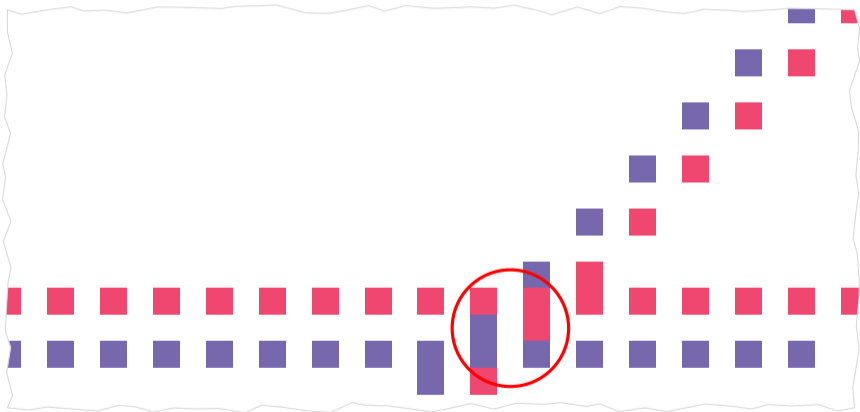
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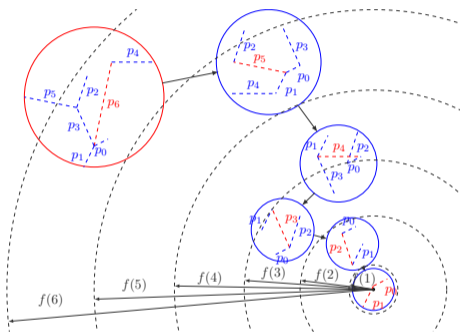
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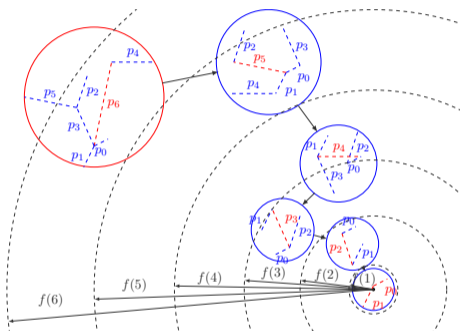
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**Corrolary 4**

[GHV18, Corollary 8]

Aperiodic Domino is co-recursively enumerable (ie.  $\Pi_1^0$ ) for  $\mathbb{Z}^2$  subshifts.

The background of the slide features a grid of light blue squares. Overlaid on this grid are several orange paths that connect the corners of the squares. These paths are arranged in a way that suggests a tiling or a specific combinatorial problem. The paths are composed of horizontal and vertical segments, with some segments being longer than others, creating a complex, non-repeating pattern.

**Aperiodic Domino problem on  $\mathbb{Z}^d, d \gtrsim 3$**

# Aperiodic Domino in higher dimension

**Input** An effective  $\mathbb{Z}^d$  subshift.

**Output** Is there an admissible *aperiodic* coloring?

**Theorem 5**

AD in high dimension is not arithmetic

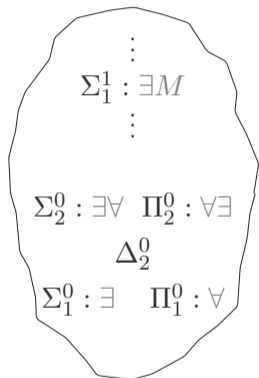
Aperiodic Domino is  $\Sigma_1^1$ -complete for:

- ▶  $\mathbb{Z}^d$  sofic subshifts with  $d \geq 3$ .
- ▶  $\mathbb{Z}^d$  SFTs with  $d \geq 4$ .

We reduce **State recurrence**:

**Input** A non-deterministic TM and a state  $q_0$

**Output** Is there a run on  $\varepsilon$  that visits  $q_0$  infinitely often?



# On $\mathbb{Z}^3$ : From AD to state recurrence (Part 1)

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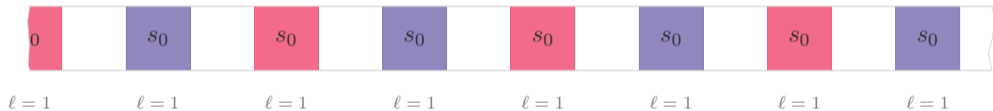


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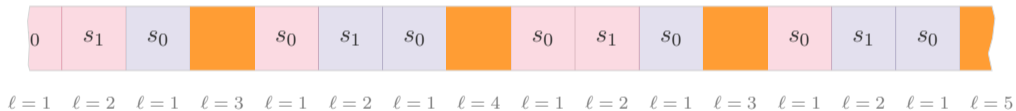
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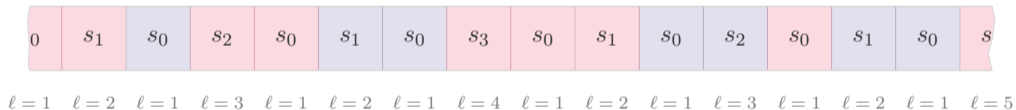
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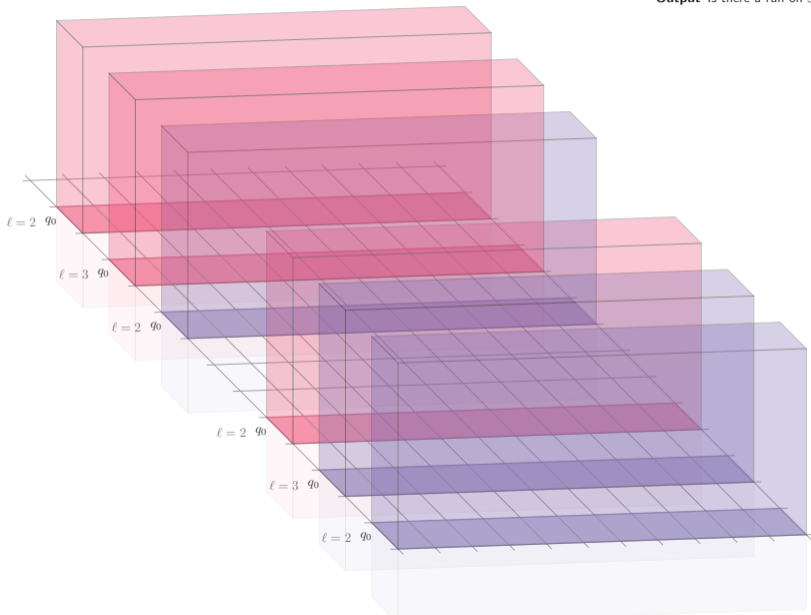
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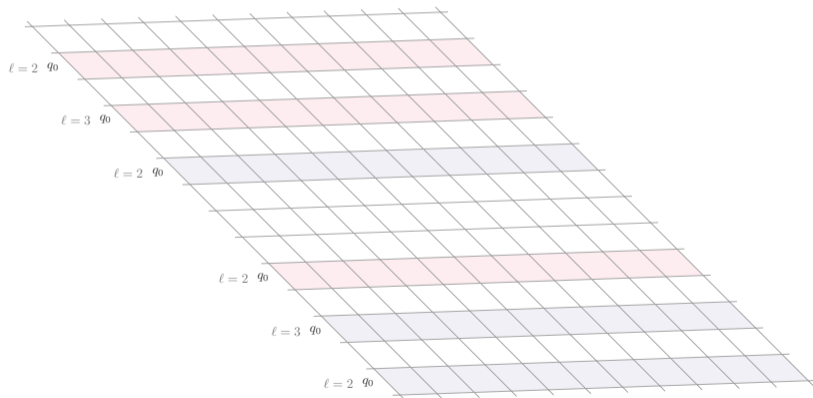


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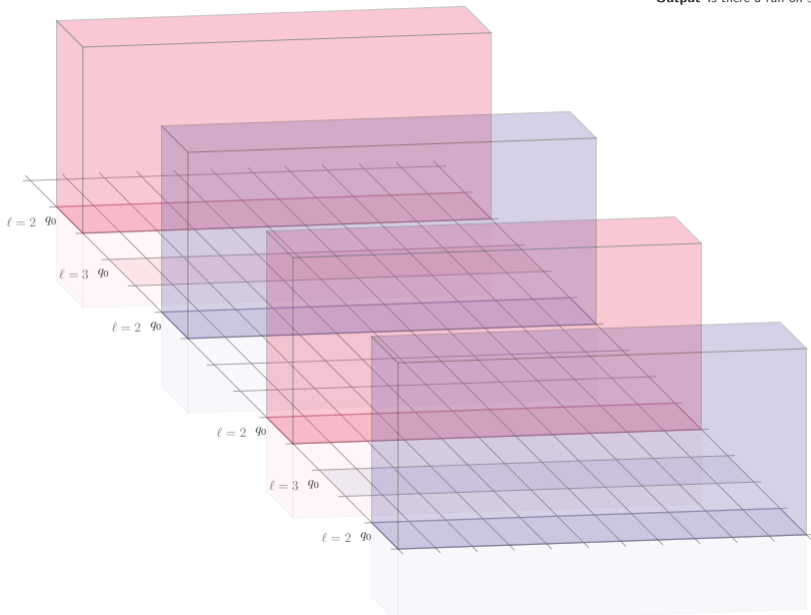


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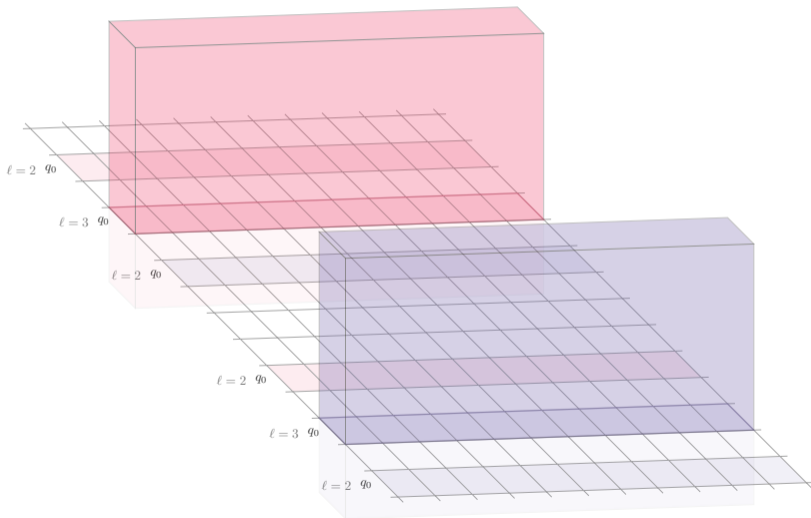
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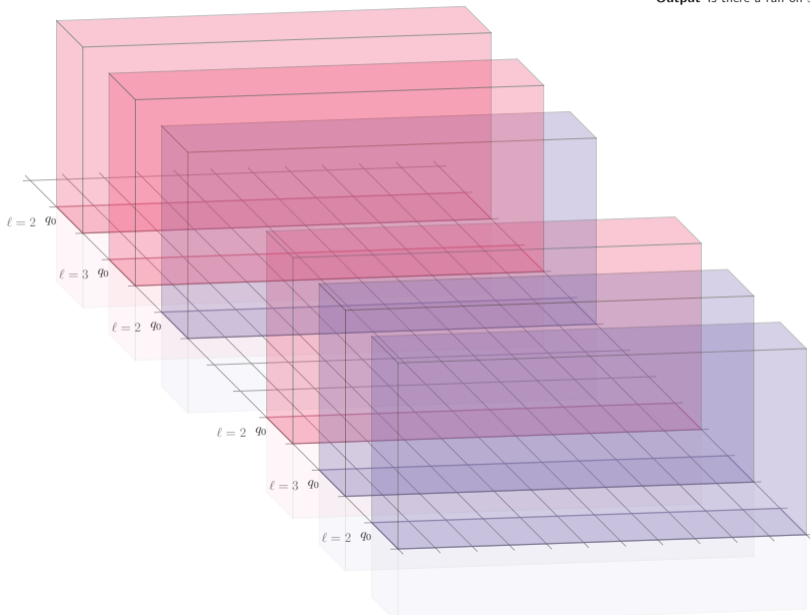
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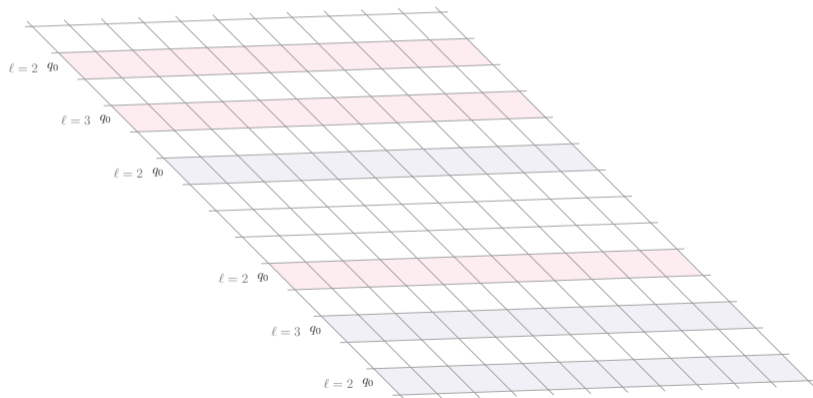


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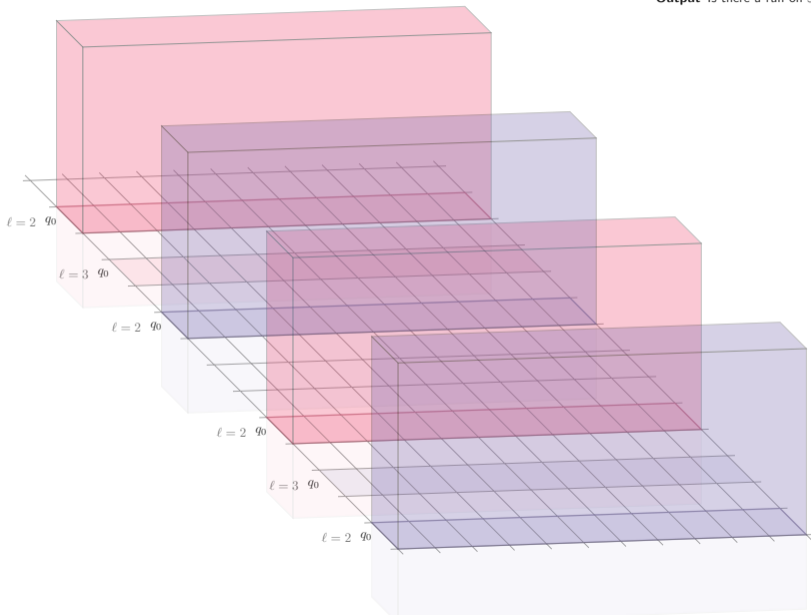


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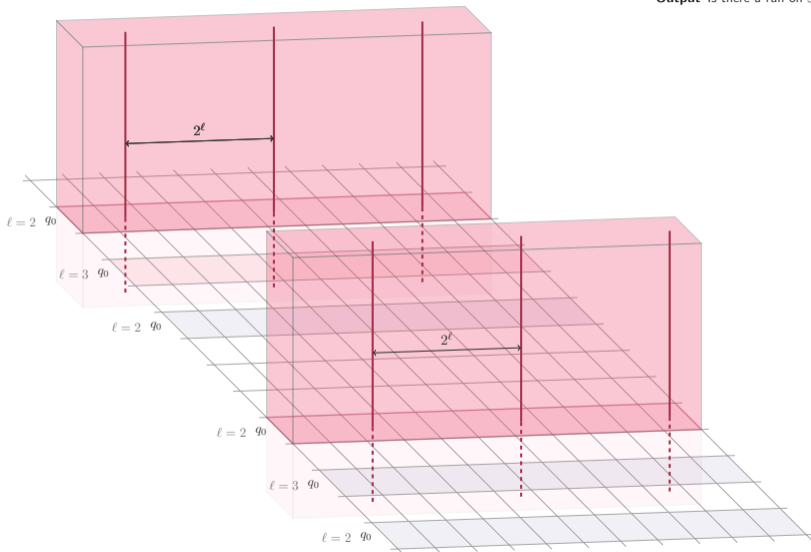


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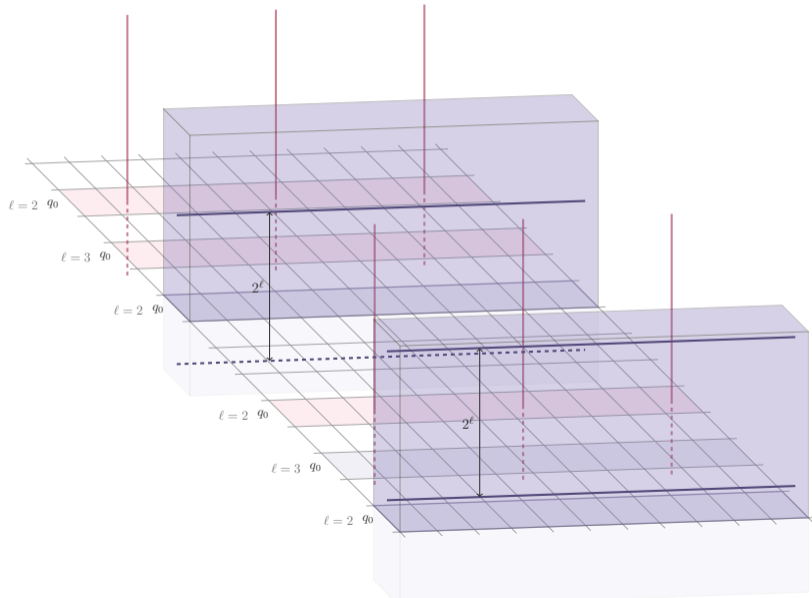


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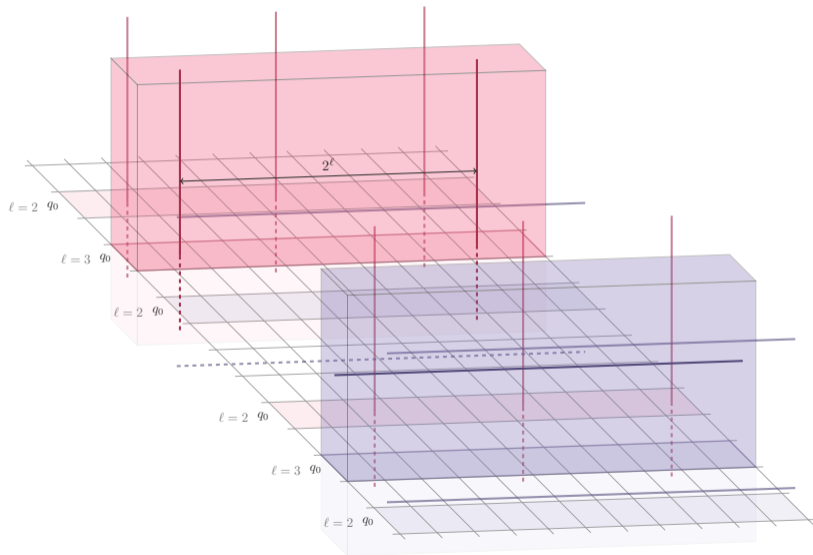


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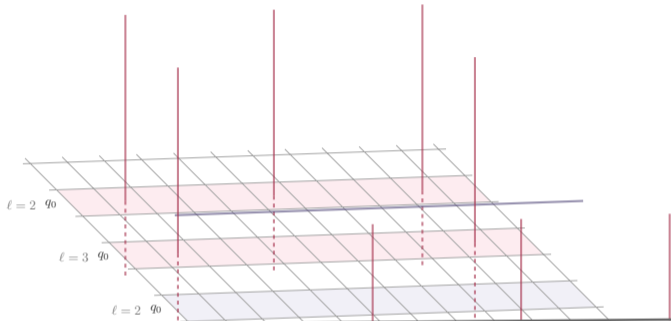


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**Aperiodic coloring  $\iff$  the machine visits  $q_0$  infinitely often.**

Why two additional dimensions? (Spoiler: I lied)

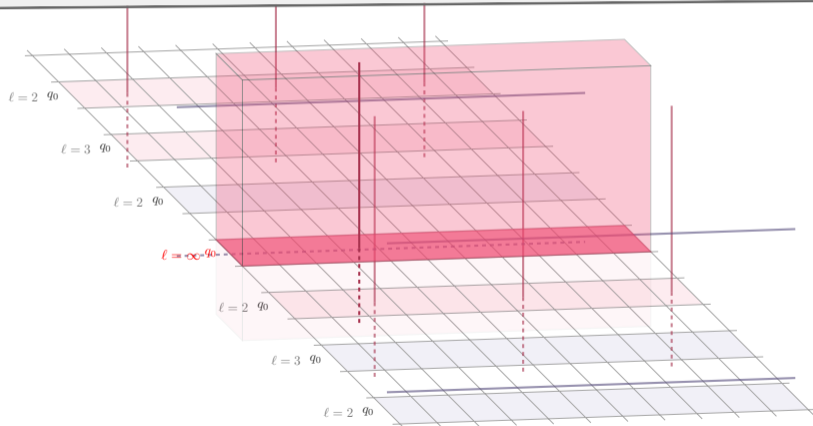
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**Beware the slice of  $\infty$  level!**



# Recap and consequences

## Aperiodic Domino problem:

**Input** An effective  $\mathbb{Z}^d$  subshift.

**Output** Is there an admissible *aperiodic* coloring?

Its (computational) complexity depends on the dimension of the subshift.

$\implies$  : separates 2, 3 and 4-dimensional subshifts.

| Dimension / type | 2D        | 3D           | 4D+          |
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| finite type      | $\Pi_1^0$ | <b>open</b>  | $\Sigma_1^1$ |
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Difficulty of the Domino problem

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Difficulty of the Domino problem

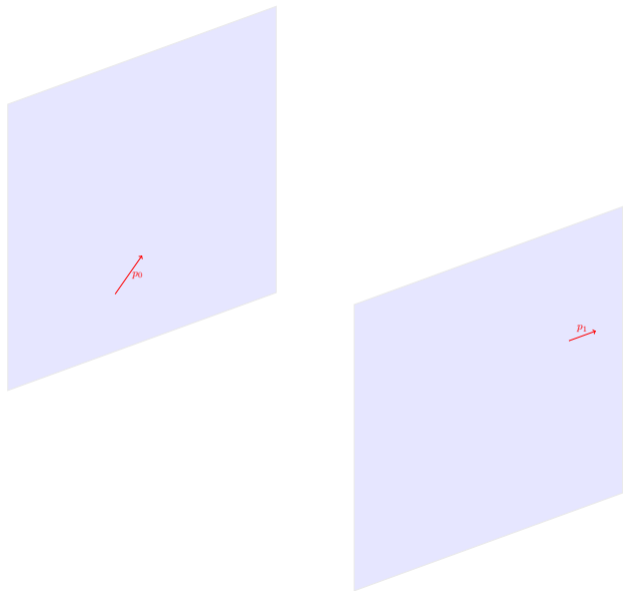
The background of the slide features a grid of light blue squares. Overlaid on this grid are orange lines that form a complex, non-repeating pattern of paths. These paths connect the corners of the squares in a way that suggests a tiling or domino problem. The paths are composed of horizontal and vertical segments, with some segments extending across multiple squares. The overall pattern is aperiodic, meaning it does not repeat in a regular, periodic fashion.

**Aperiodic Domino problem on  $\mathbb{Z}^3$  SFTs**

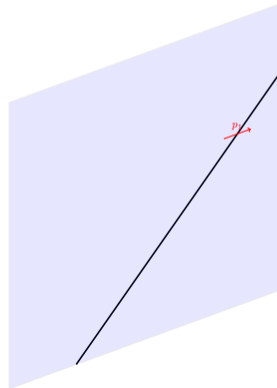
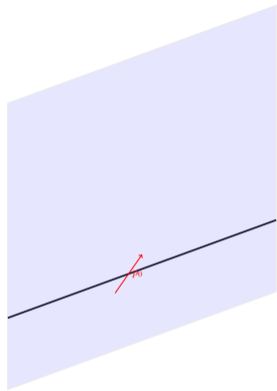
Two lines must cross in  $\mathbb{Z}^3$  SFTs



## Two lines must cross in $\mathbb{Z}^3$ SFTs

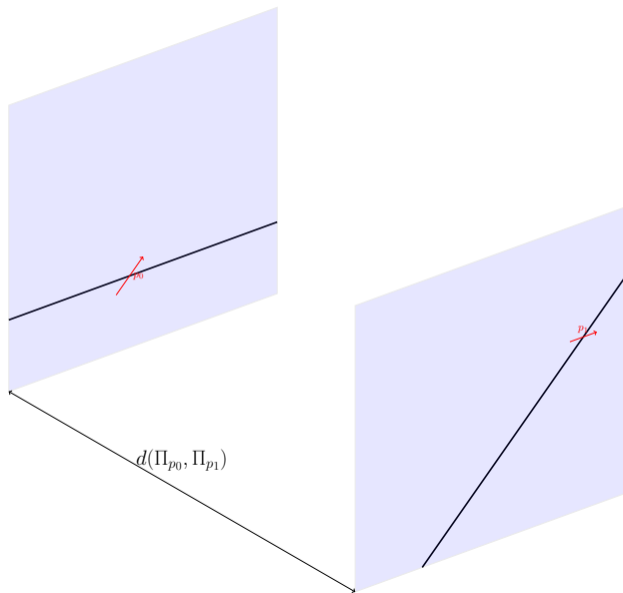


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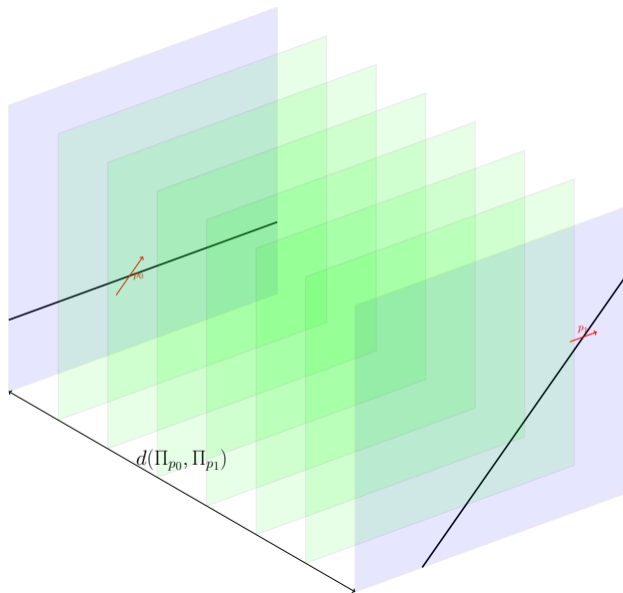




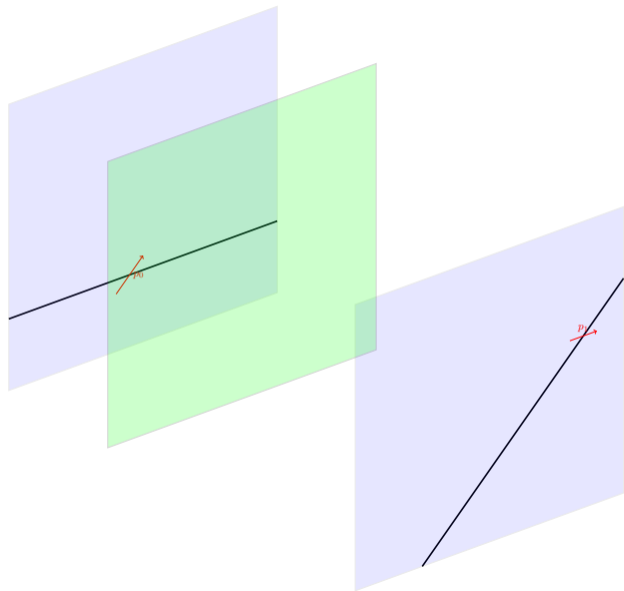
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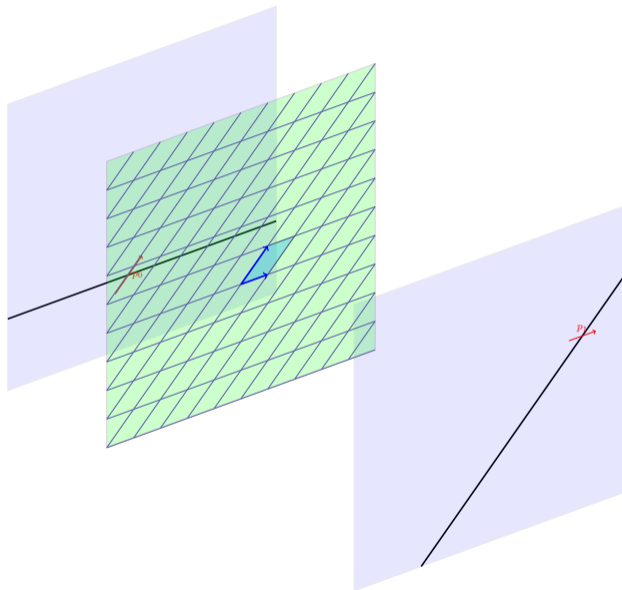
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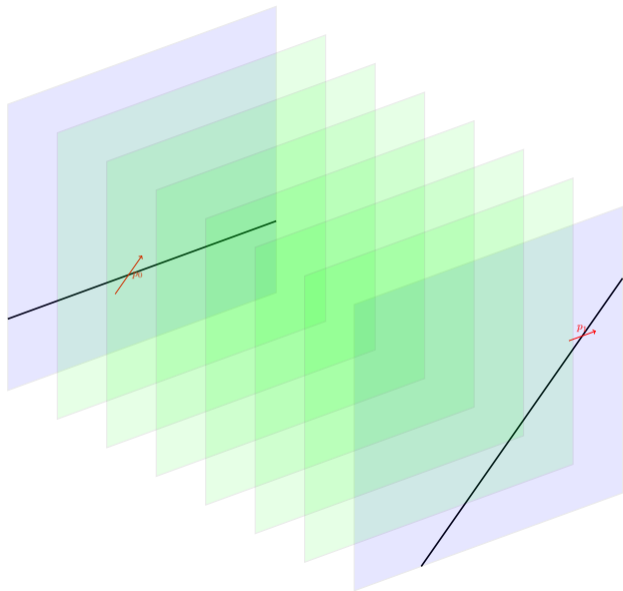
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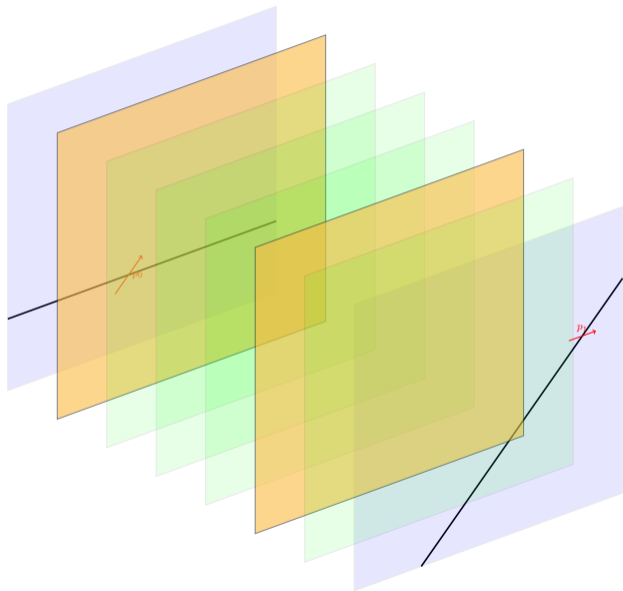
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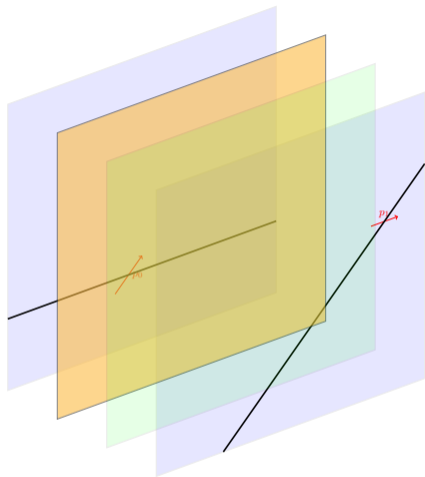
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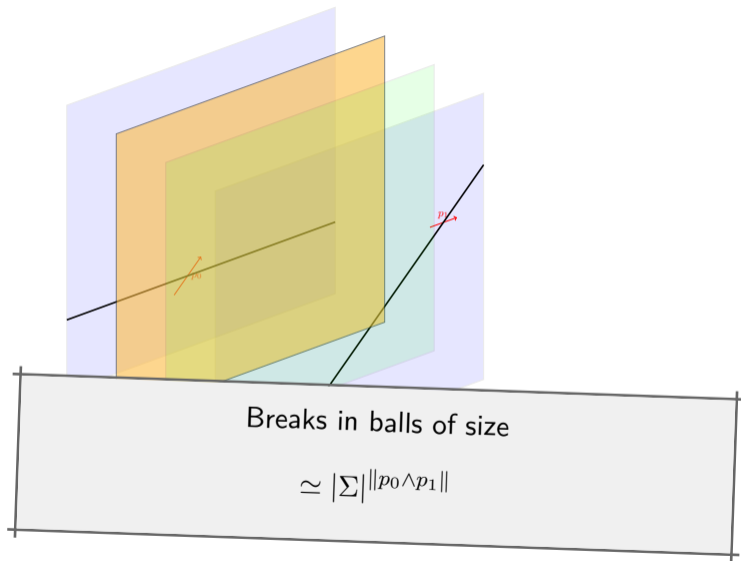
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# AD on $\mathbb{Z}^3$ SFTs

Do  $\mathbb{Z}^3$  SFTs behave similarly to  $\mathbb{Z}^2$  subshifts in terms of aperiodicity? i.e.

## Conjecture 6

There exists a computable function  $f$  such that any  $\mathbb{Z}^3$  SFT  $X \in \mathbf{EAC}$  contains **a configuration which breaks any period**  $\|p\| \leq n$  inside the centered square of edge  $f(W, |\Sigma|, n)$ .

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Why?

- ▶ Two lines of period breaks do not cross on  $\mathbb{Z}^3$  in general... But they do in  $\mathbb{Z}^3$  SFTs.
- ▶ Embedding computations in SFTs requires two dimensions.
- ▶ By compactness (i.e. limits), the two additional dimensions of the  $\Sigma_1^1$ -proof *are necessary*.

The background of the slide features a repeating pattern of light blue squares arranged in a grid. Overlaid on this grid are orange lines that form a complex, non-periodic path, resembling a Hamiltonian path or a similar combinatorial structure. The lines connect the corners of the squares in a way that does not repeat periodically across the plane.

## **Aperiodic Domino problem and entropies**

# Complexity function

## Definition 7

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The *complexity function*  $N_X(n)$  is defined as the number of different patterns of size  $n^d$  that appear in  $X \subseteq \Sigma^{\mathbb{Z}^d}$ .

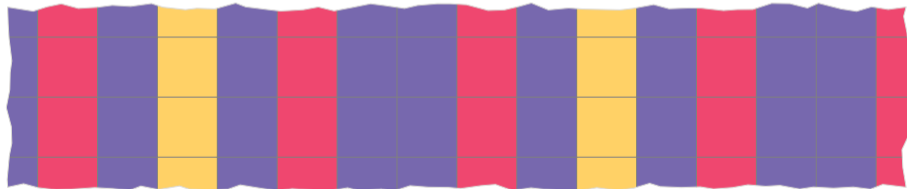
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Example : the SFT  $X$  defined by



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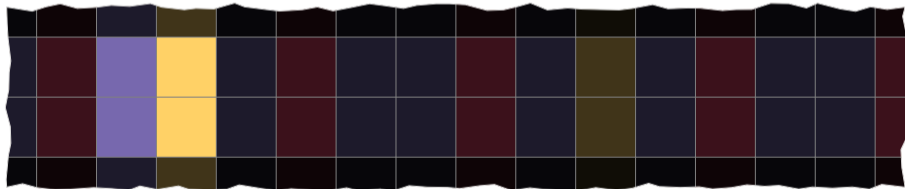
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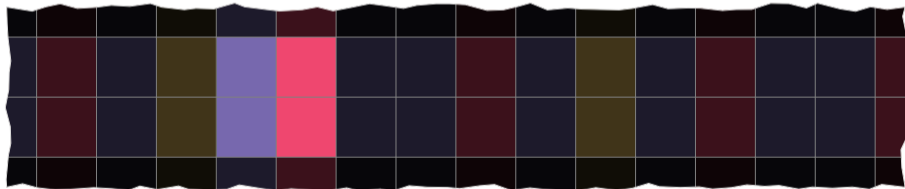
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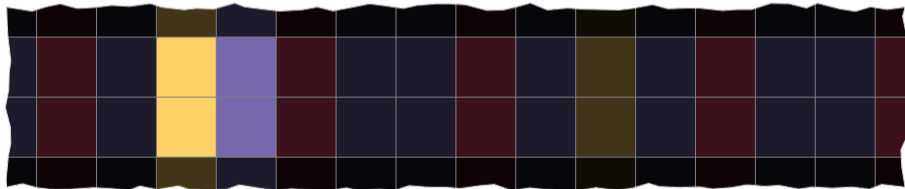
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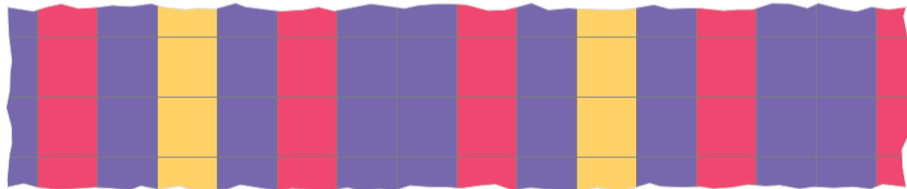
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Define the  $m$ -entropy:

$$h_m(X) = \limsup_{n \rightarrow +\infty} \frac{\log N_X(n)}{n^m}$$

## Theorem 8

Improves [Hochman, 2009]

Let  $X$  be a  $\mathbb{Z}^d$  subshift. If  $h_{d-1}(X) = +\infty$ , then there exists an aperiodic configuration in  $X$ .

# Aperiodic Domino and entropies

For  $X$  a  $\mathbb{Z}^d$  subshift,  $h_{d-1}(X) = \limsup_{n \rightarrow +\infty} \frac{\log N_X(n)}{n^{d-1}}$  implies that  $X$  contains an aperiodic configuration.

Proof:

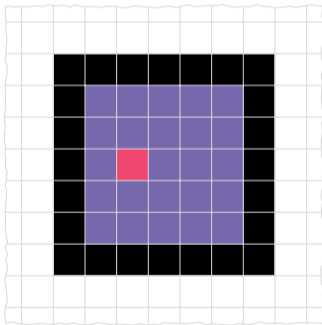
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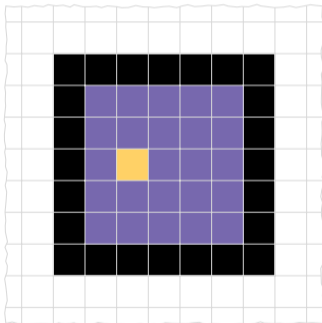


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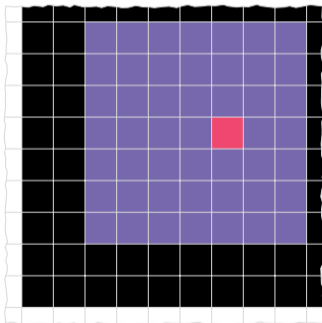


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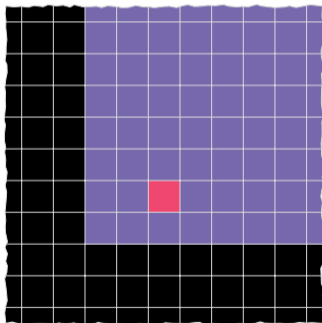


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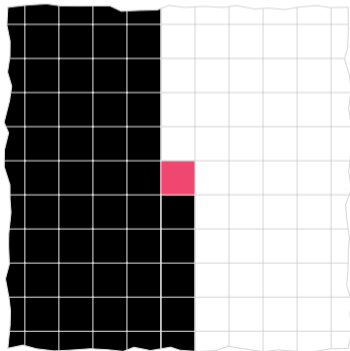


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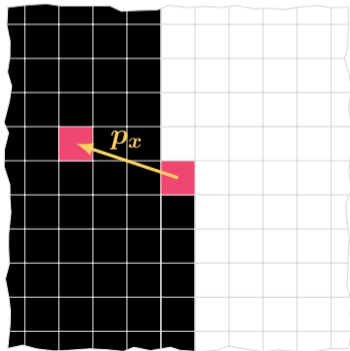


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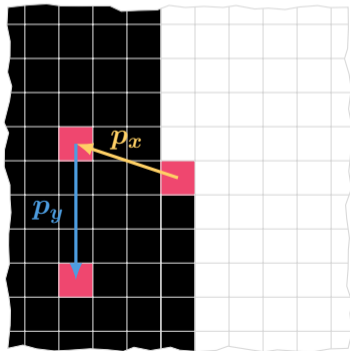


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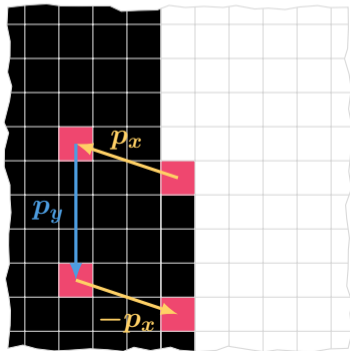


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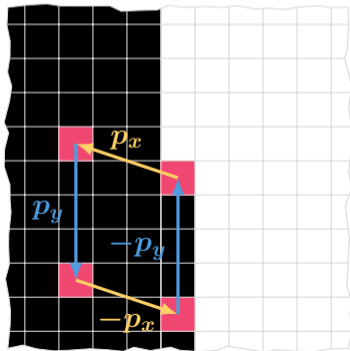


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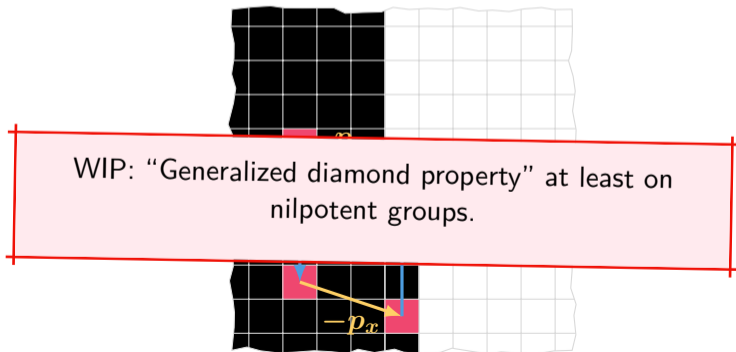


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# Conclusion

## Aperiodic Domino problem:

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**Output** Is there an admissible *aperiodic* coloring?

## Computational complexity:

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Difficulty of the Domino problem

## Relates to entropies:

For  $X$  a  $\mathbb{Z}^d$  subshift, if  $h_{d-1}(X) = +\infty$ , then  $X$  contains an aperiodic configuration.



Thank you

Questions?

école \_\_\_\_\_  
normale \_\_\_\_\_  
supérieure \_\_\_\_\_  
paris—saclay \_\_\_\_\_

