Pattern complexity, aperiodic domino problem and group geometry

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Université de Caen Normandie

Journées SDA2,
Toulouse, March 29th 2023

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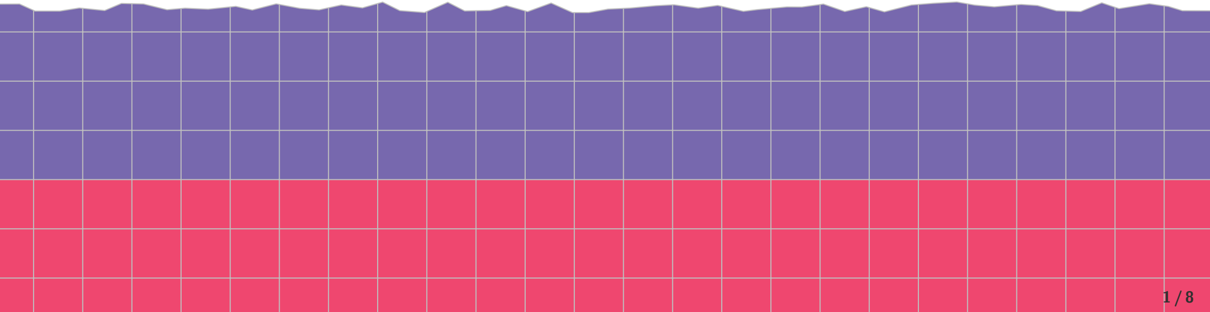
Subshifts

Definition 1 Subshifts

A $\mathbb{Z}^d$ *subshift* is a set of colorings $\mathbb{Z}^d \mapsto \Sigma$ defined by forbidden patterns $\mathcal{F}$:

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

$\Sigma = \{$  $\}$ and $\mathcal{F} = \{$  $\}$



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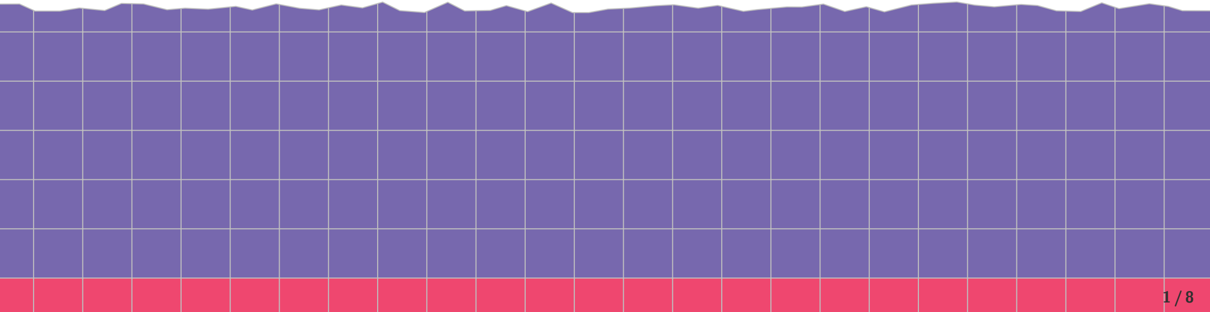

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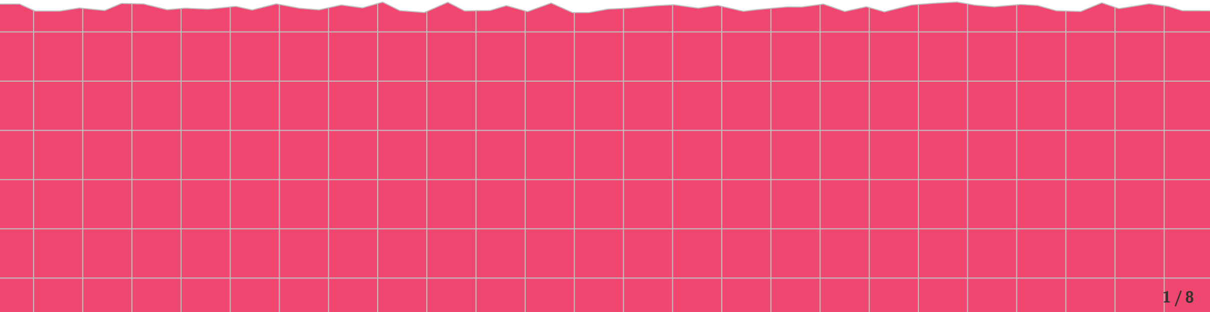

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The family $\mathcal{F}$ can be either finite (X is *of Finite Type*), or infinite.

Classical Domino problem:

Input A SFT (= symbols + finite set of forbidden pattern)

Output Is there an admissible coloring?

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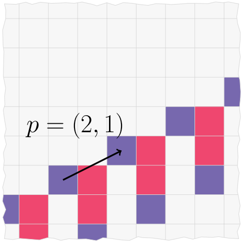

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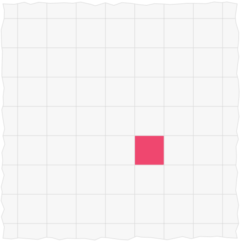

Aperiodic Domino problem on $\mathbb{Z}^d$

Undecidability of the aperiodic Domino problem

Aperiodic Domino problem:

Input A $\mathbb{Z}^d$ subshift.

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Its (computational) complexity depends on the dimension of the subshift.

$\implies$: separates $2, 3$ and 4-dimensional subshifts.

Dimension / type	2D	3D	4D+
finite type	Π_1^0	**open**	Σ_1^1
sofic	Π_1^0	Σ_1^1	Σ_1^1
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Difficulty of the Domino problem [GHV 2018, CH 2022].

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Surface entropy and aperiodic domino

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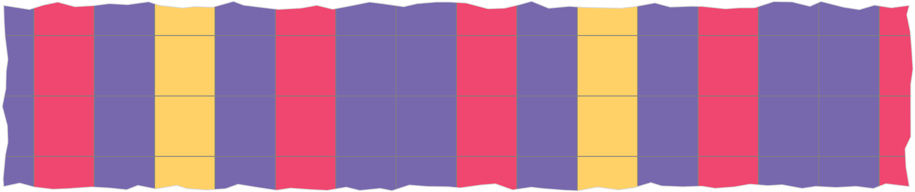

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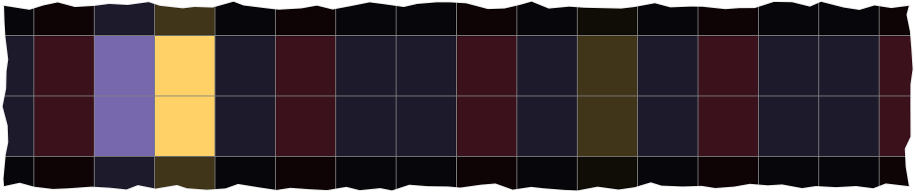

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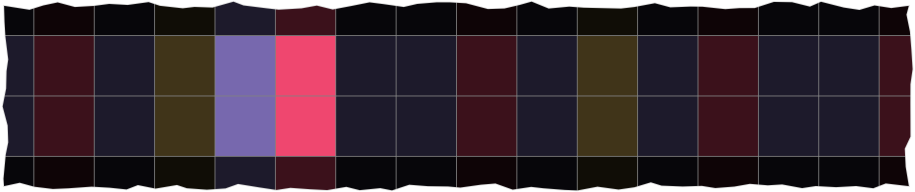

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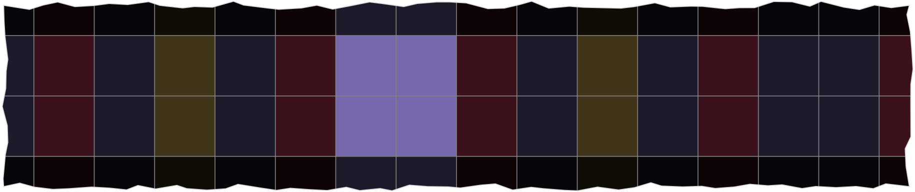

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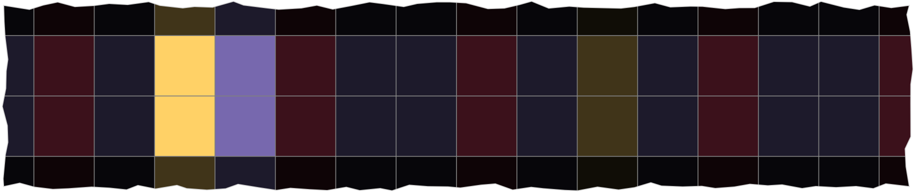

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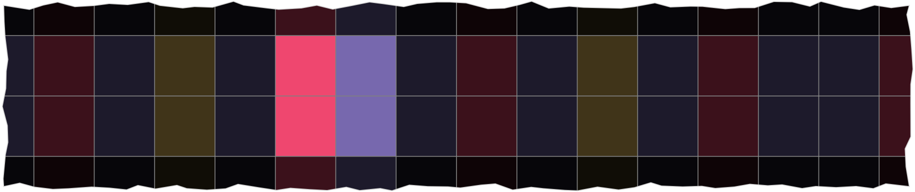

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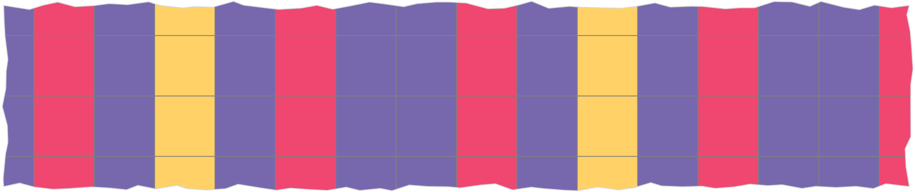

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Define the surface entropy of X:

$$h_{d-1}(X) = \limsup_{n \to +\infty} \frac{\log N_X(n)}{n^{d-1}}$$

Theorem 3 [C. Hellouin, Salo]

Let X be any $\mathbb{Z}^d$ subshift. If $h_{d-1}(X) = +\infty$, then there exists an aperiodic configuration in X.

Proof.

For X a $\mathbb{Z}^d$ subshift. Assume $h_{d-1}(X) = \limsup_{n\to+\infty} \frac{\log N_X(n)}{n^{d-1}}$.

Claim 1: For every $k \in \mathbb{N}$, there exists a border of thickness k with at least two admissible completions of its interior.

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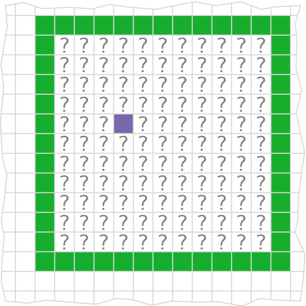
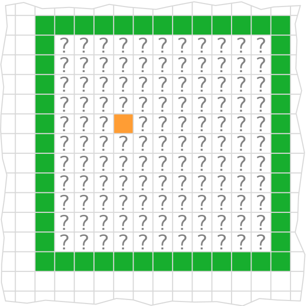
Claim 1: For every $k \in \mathbb{N}$, there exists a border of thickness k with at least two admissible completions of its interior. Indeed,

$$\#X|_{\partial_k [\![0,n]\!]^d} \leq |\Sigma|^{2d\cdot k\cdot n^{d-1}} \qquad \text{and} \qquad \#X|_{[\![0,n]\!]^d} = N_X(n)$$

As $\log N_X(n)/n^{d-1}$ is unbounded, there exists $n \in \mathbb{N}$ such that

$$\#X|_{\partial_k [\![0,n]\!]^d} < \#X|_{[\![0,n]\!]^d}$$





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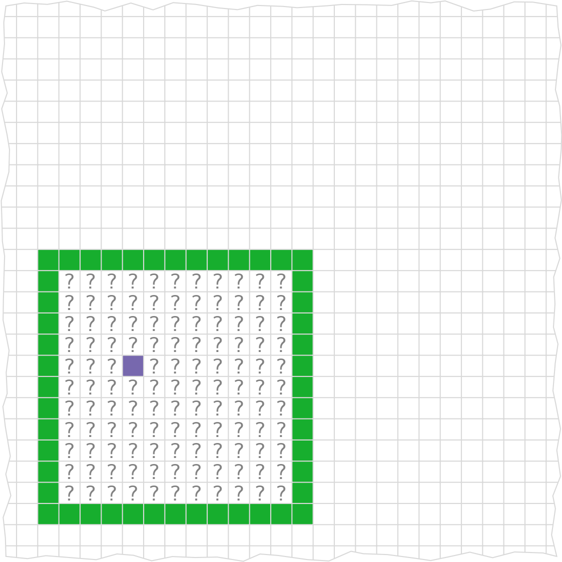

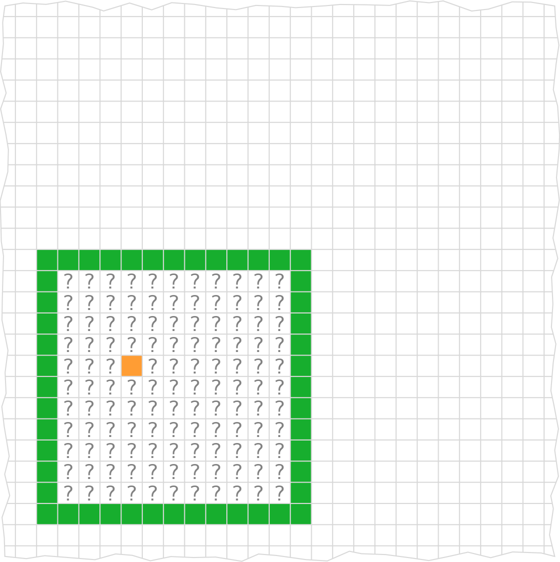

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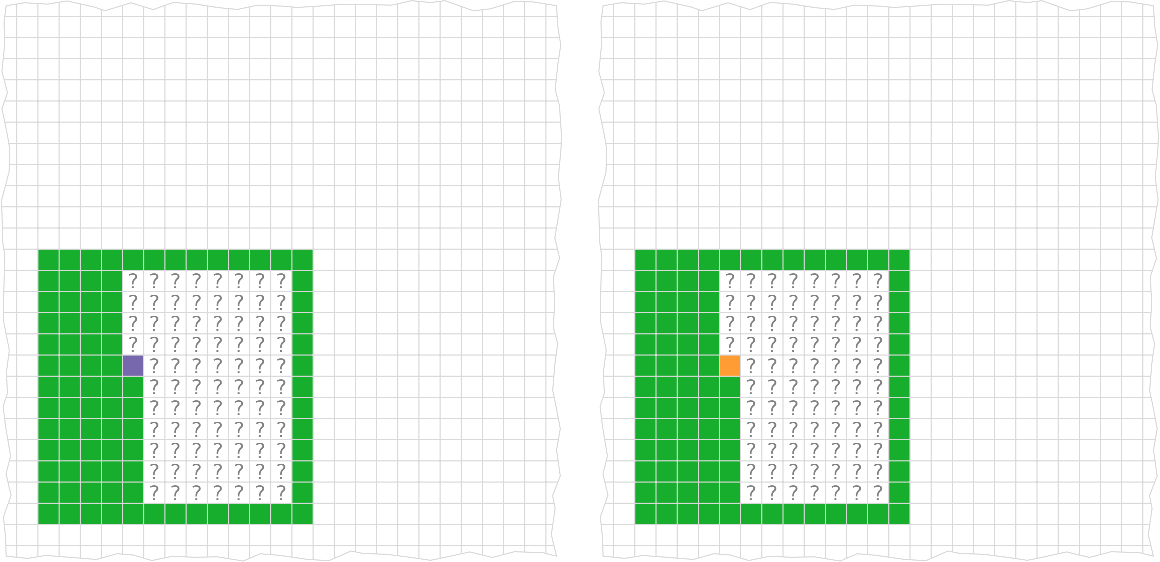

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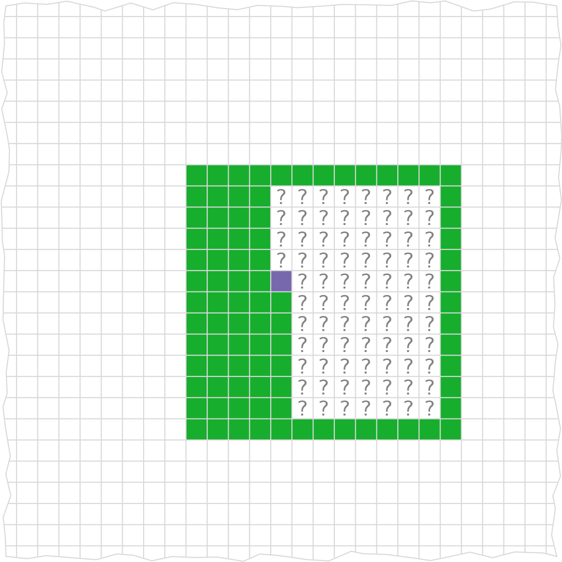

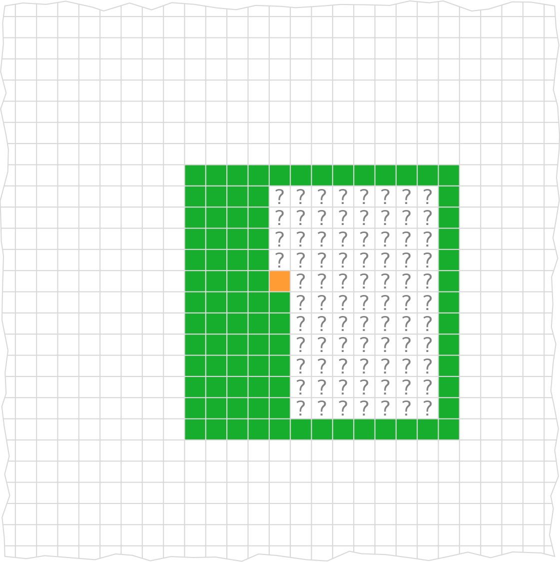

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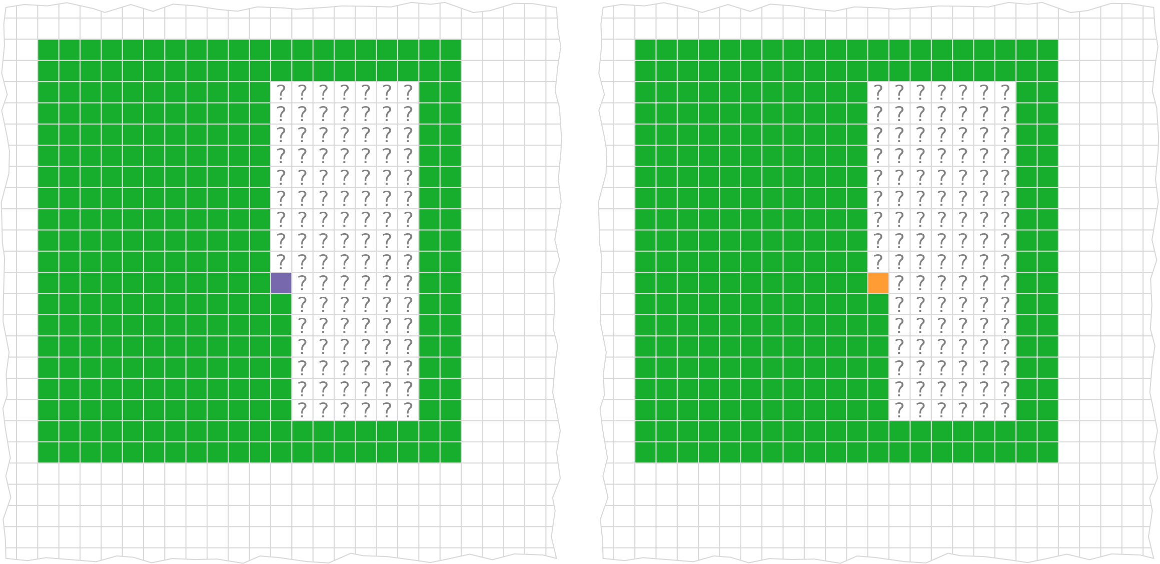

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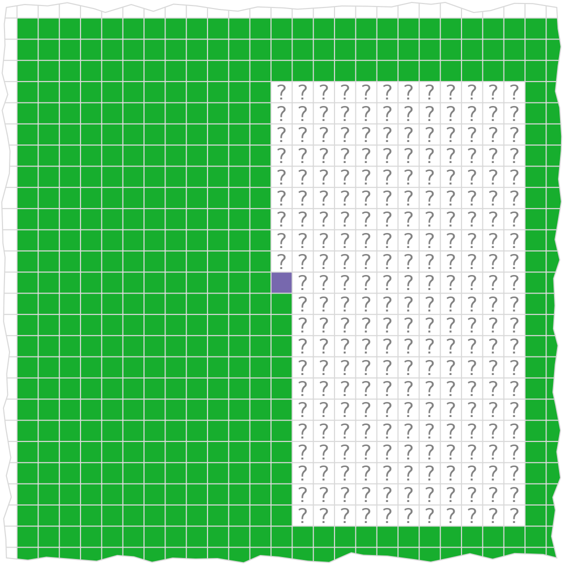

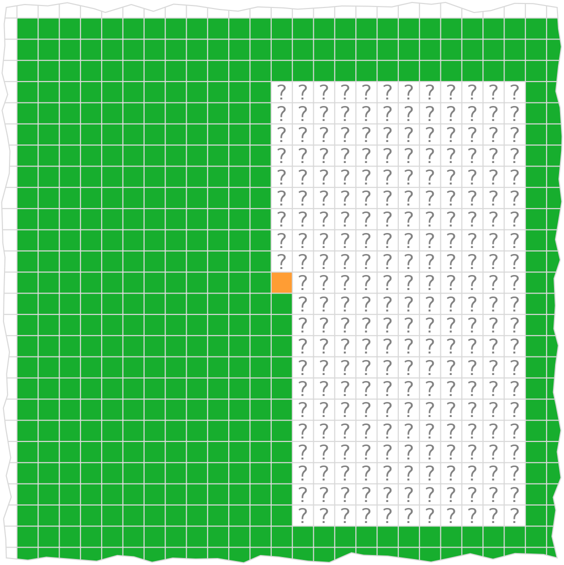

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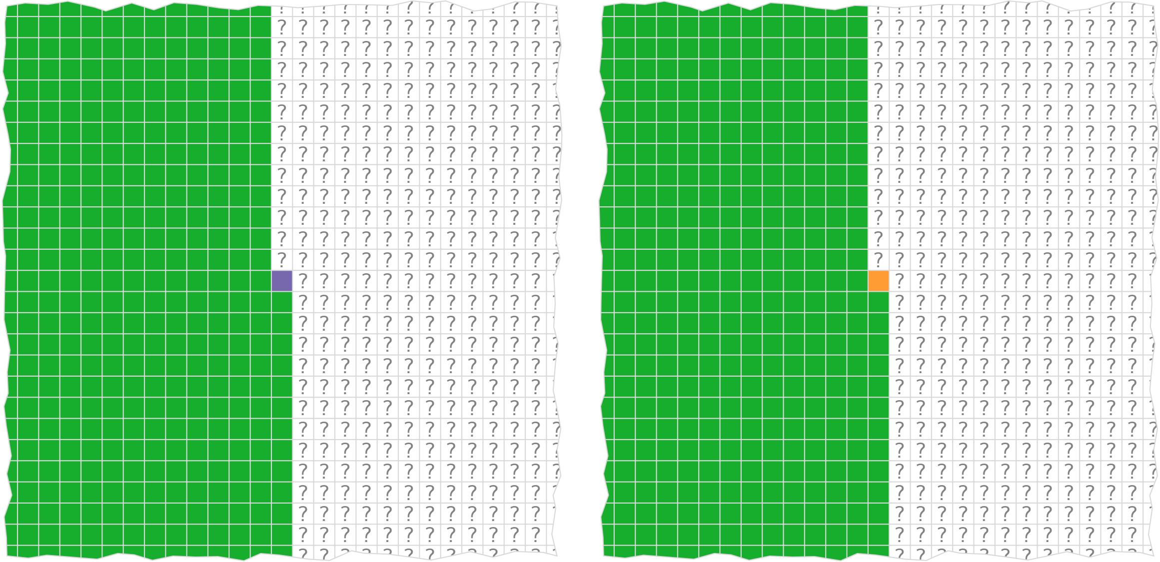
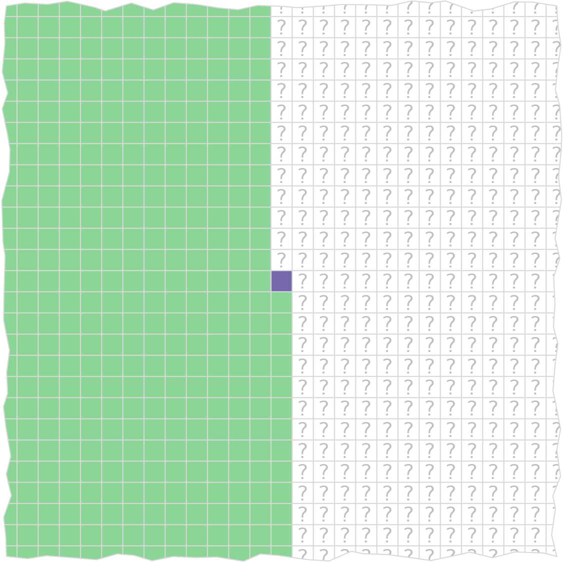
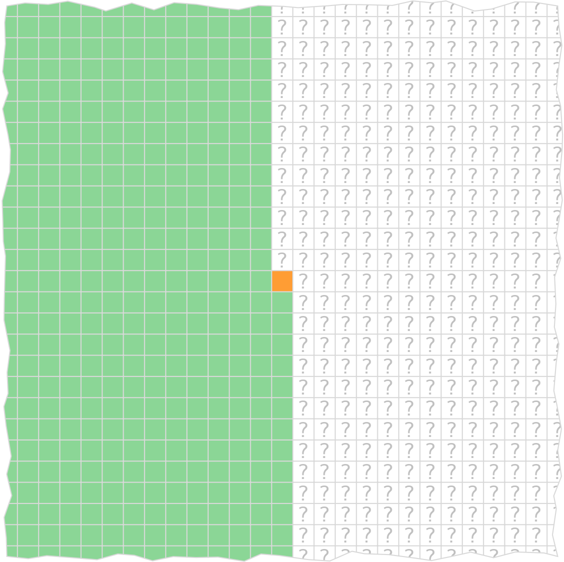


Proof. Assume p_l and p_r are respective non-trivial periods for these configurations.





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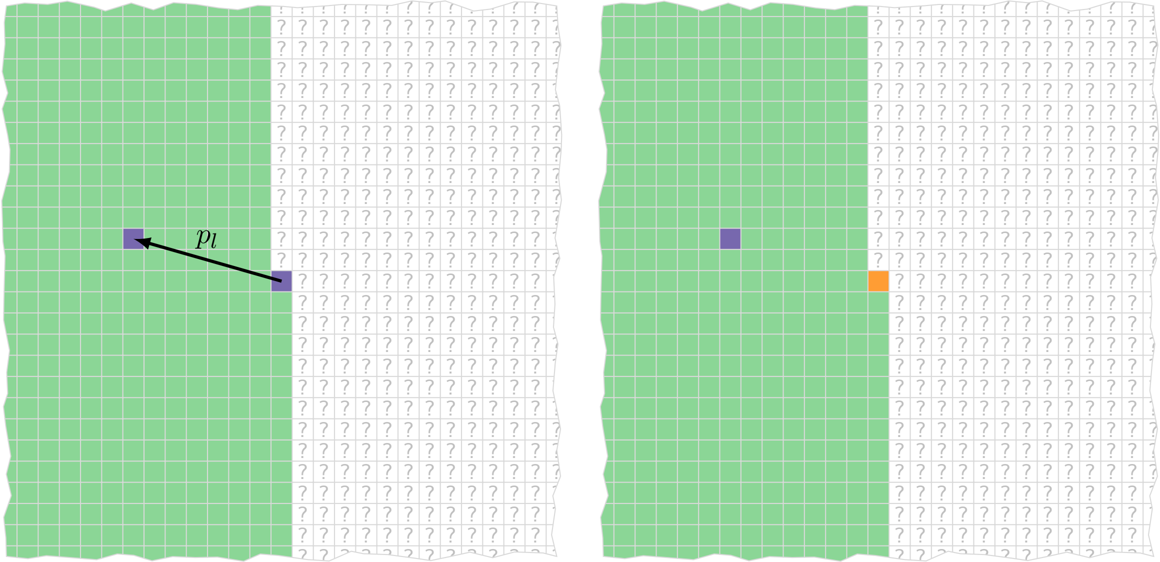

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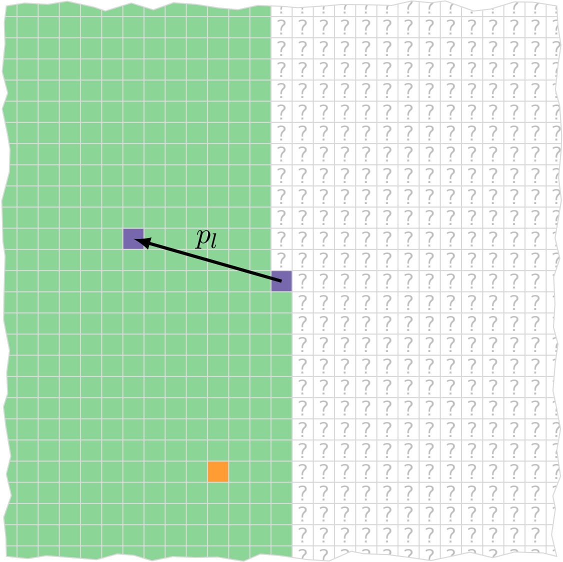

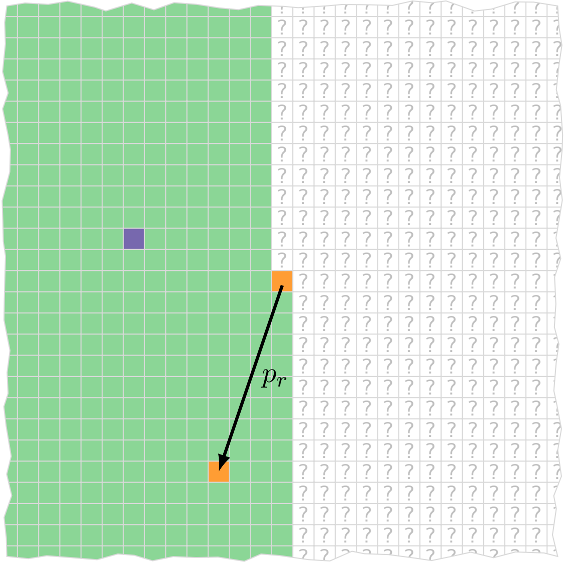

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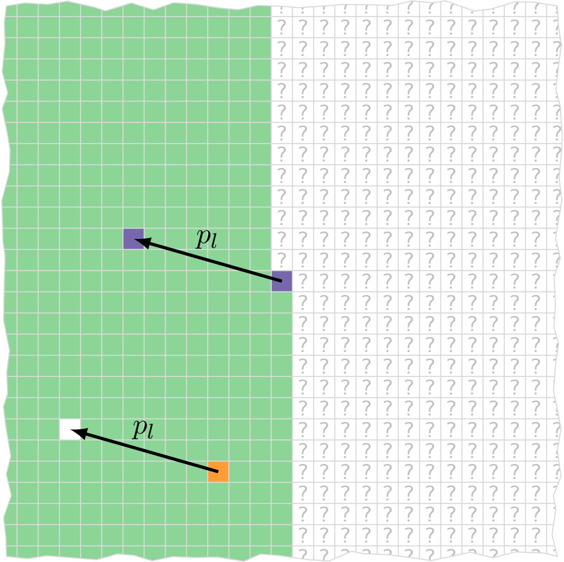

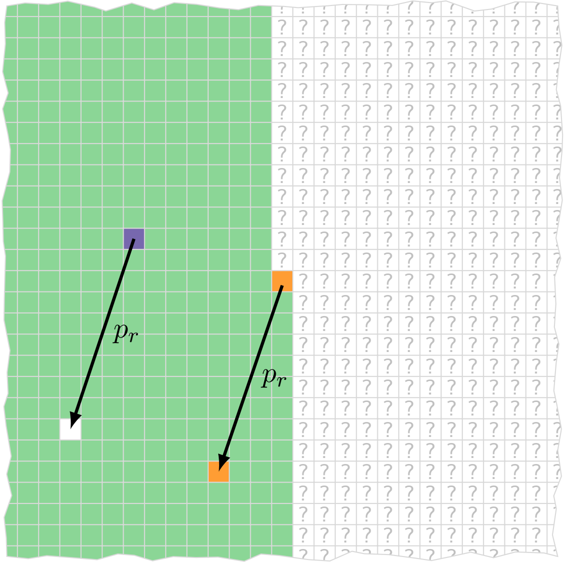

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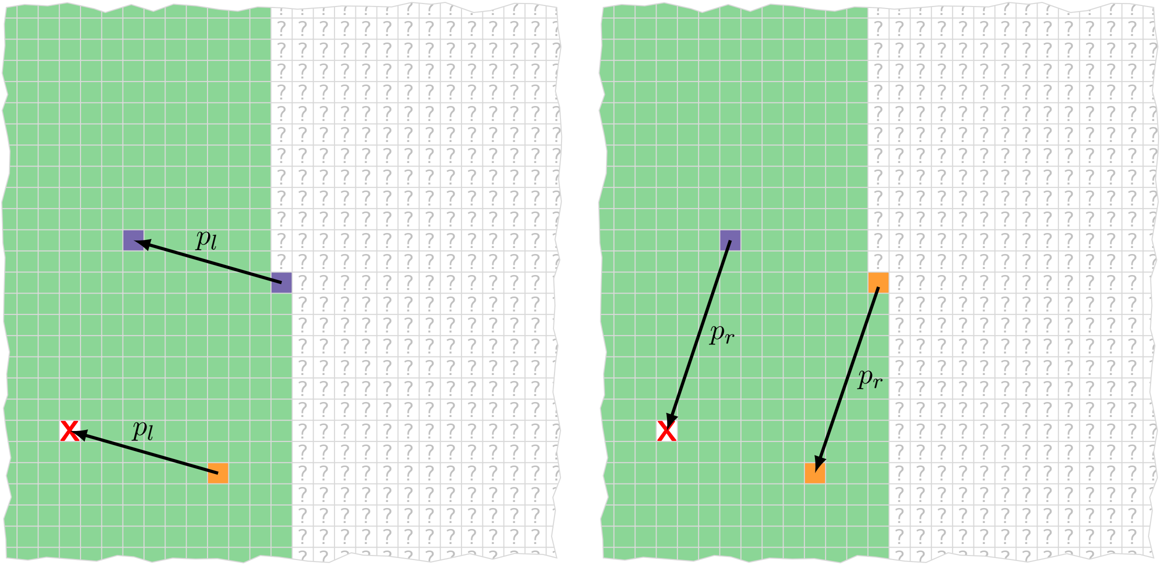

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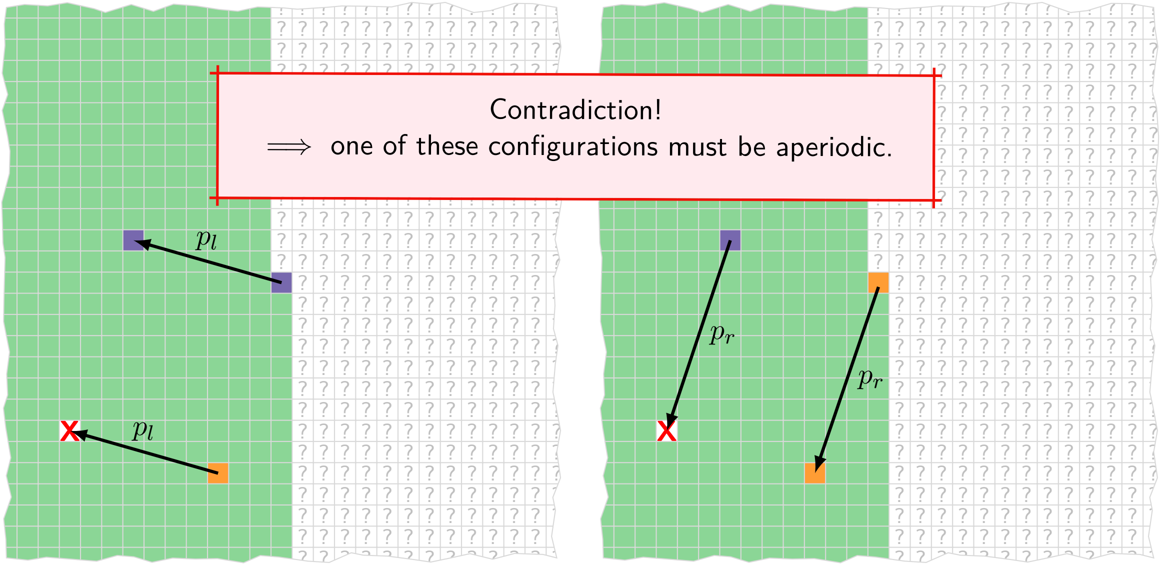

Contradiction!

$\implies$ one of these configurations must be aperiodic.

On other groups than $\mathbb{Z}^d$?

Let $(G, \leq)$ be an ordered group.

Definition 4 Confluence

$(G, \leq)$ is *confluent* if for any two vectors $a, b < 1_G$, a and b can be joined in the negative semi-lattice generated by a and b.

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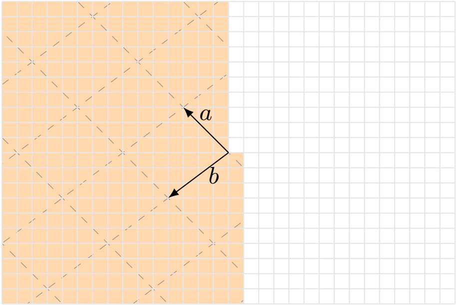

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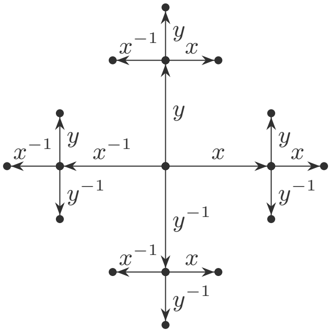

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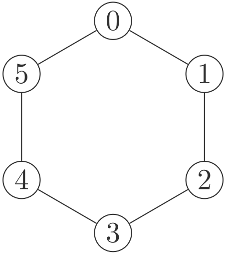

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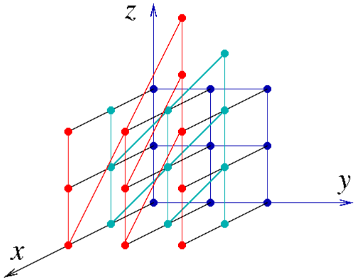

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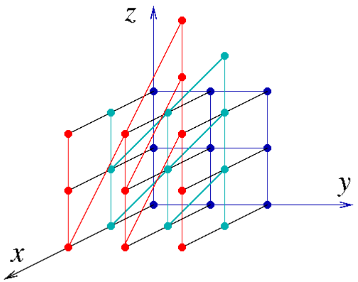
So (torsion-free) abelian groups are confluent.
$\mathcal{H}$ (nilpotent) is confluent.



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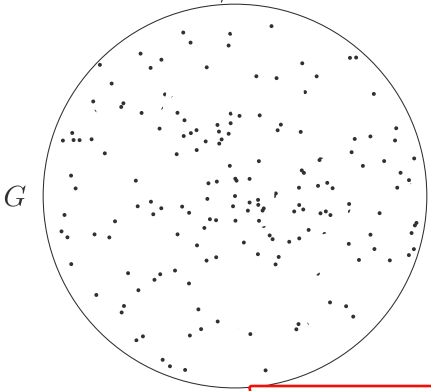

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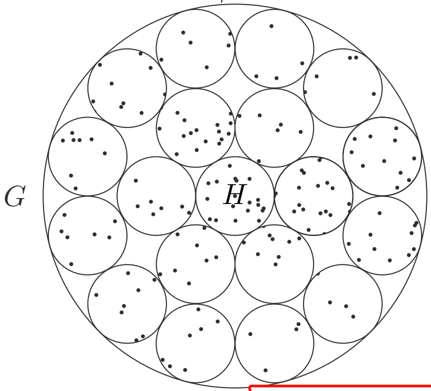

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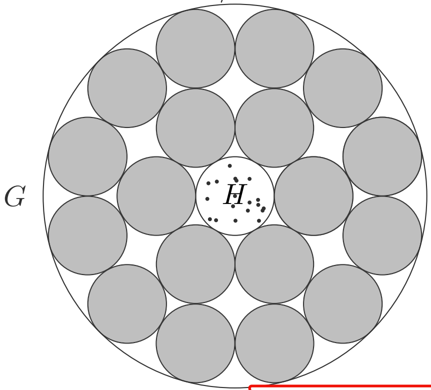

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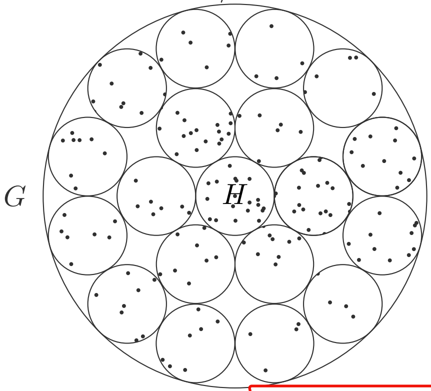

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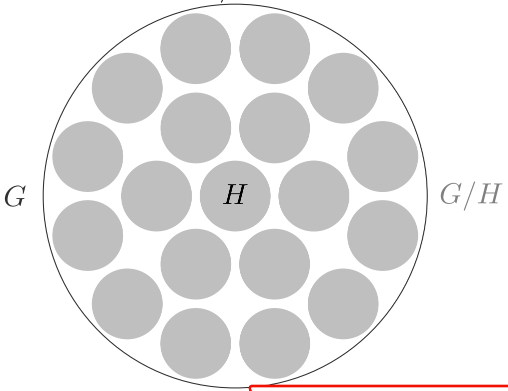

Theorem 5 Stability by group extension

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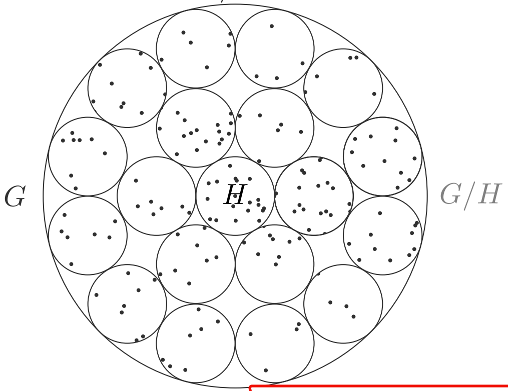

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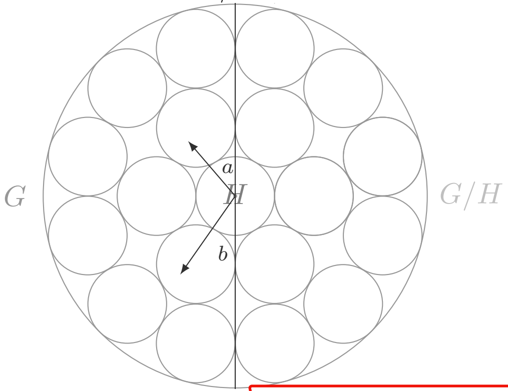

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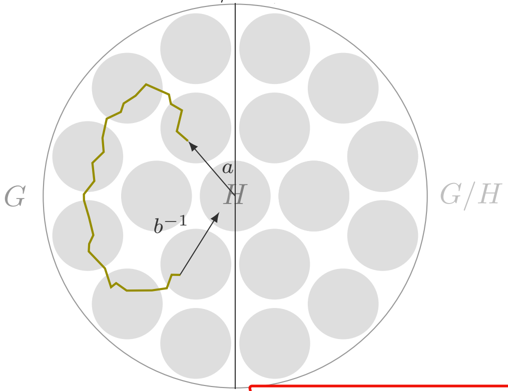

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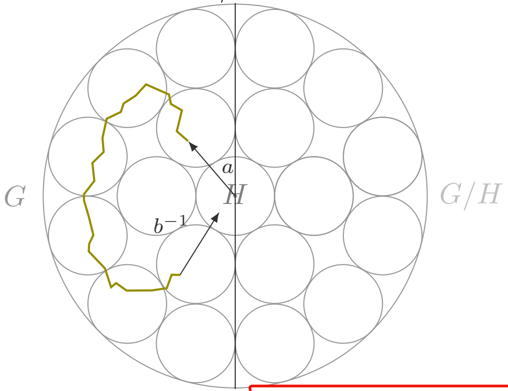

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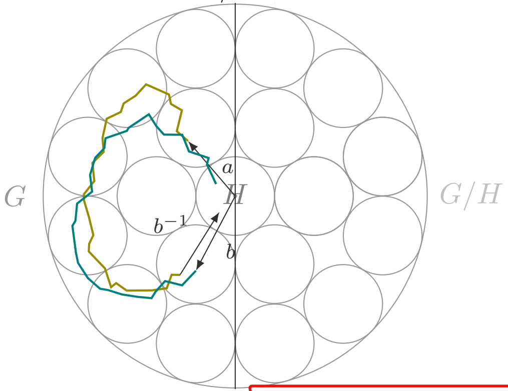

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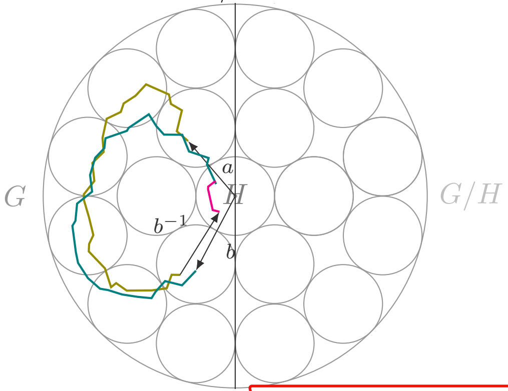

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By confluence in H, there exists W such that AWB^{-1} joins A and B in H^-.

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Corrolary 6

Poly–(torsion-free abelian) ordored groups are confluent.

Theorem 7

If $X \subseteq \Sigma^G$ has "infinite surface entropy" and $(G, \leq)$ is confluent, then X contains an aperiodic configuration.

Conclusion

Aperiodic Domino problem:

Input An effective $\mathbb{Z}^d$ subshift.

Output Is there an admissible *aperiodic* coloring?

Relates to entropies:

For X a $\mathbb{Z}^d$ subshift, if $h_{d-1}(X) = +\infty$, then X contains an aperiodic configuration.

The proof generalizes to groups, and relies on the notion of *confluence*:

$(G, \leq)$ is *confluent* if for any two vectors $a, b < 1_G$, a and b can be joined in the negative semi-lattice generated by a and b.


That's all Folks!