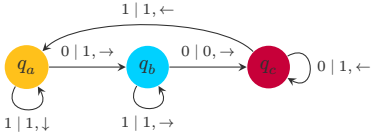
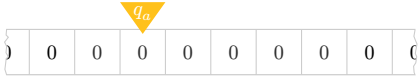
Fast computations in higher-dimensional tilings

Antonin Callard

Journées SDA2 2025 – Montpellier

Turing machines





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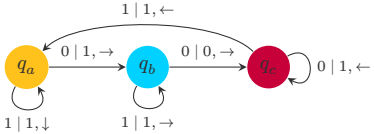

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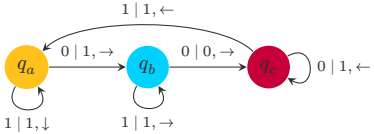

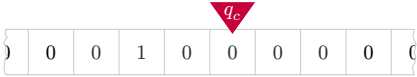

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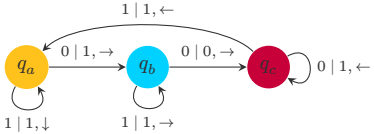

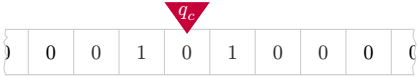

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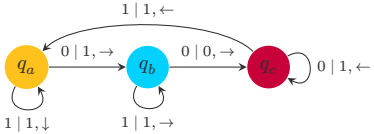

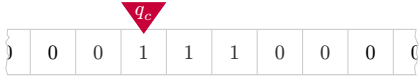

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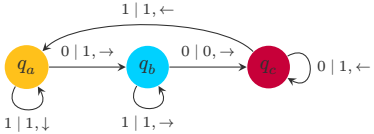

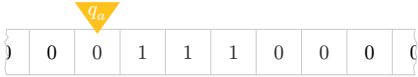

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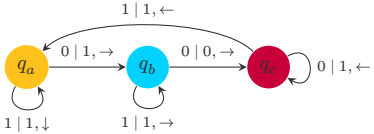

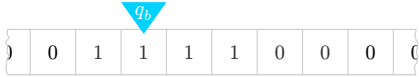

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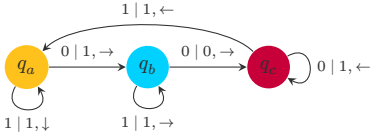

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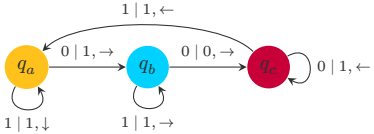

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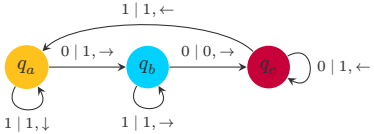

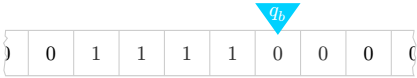

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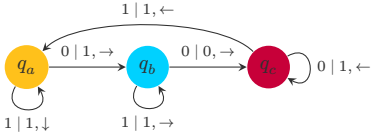

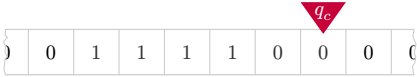

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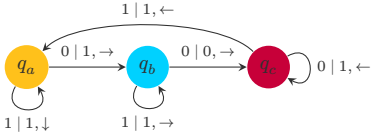

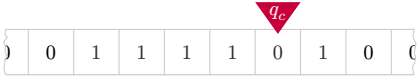

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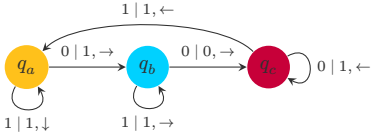

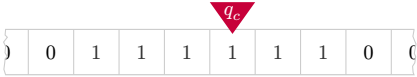

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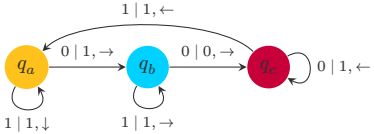

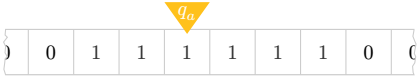

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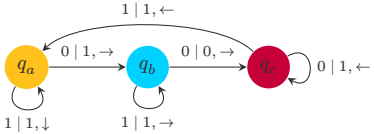

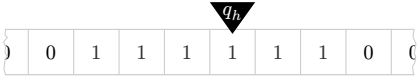

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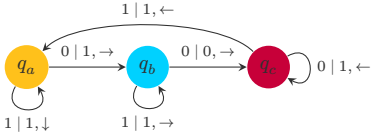
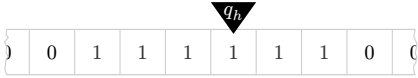
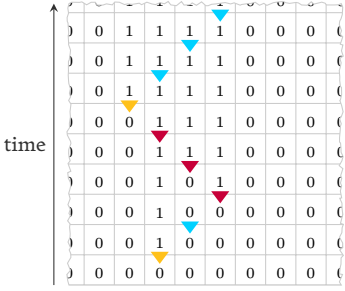
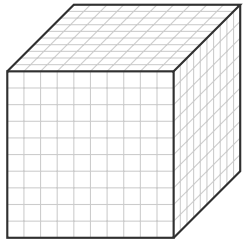




In 2-dimensional tilings, we can embed arbitrary computations by drawing *space-time diagrams*:





Main question

But what about higher dimensions? Space/time tradeoff?

Mesh-Connected MultiComputers

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Definition

A *Mesh-Connected MultiComputer* is an array of *processors*.

A processor:

- Contains finitely many *variables*;
- Can perform arithmetic operations on these variables;
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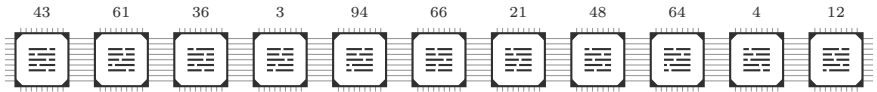
Consider the SORTING problem:

SORTING
Input: An array a of integers
Output: The array a sorted in increasing order

a =

43	61	36	3	94	66	21	48	64	4	12

Mesh-Connected MultiComputers



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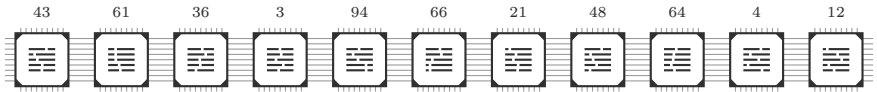

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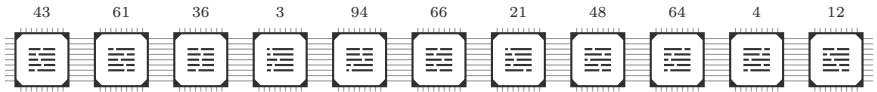

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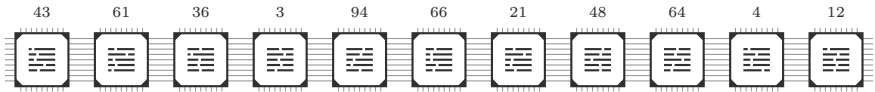

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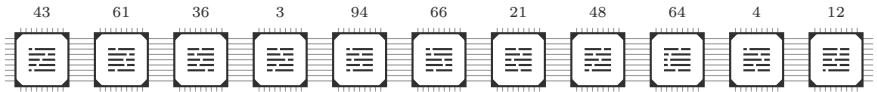

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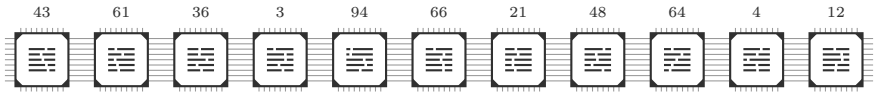

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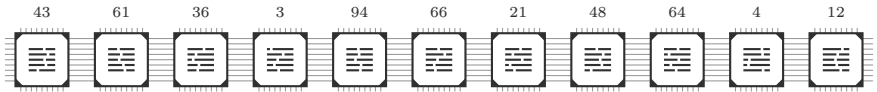

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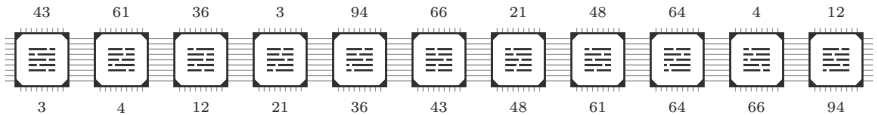

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Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
    if k is even **then**
      Compare (and swap if needed) every even position with its right neighbor;
    else (k is odd)
      Compare (and swap if needed) every odd position with its right neighbor.

3	62	50	42	67	13	57	66	47	93	92	48	50

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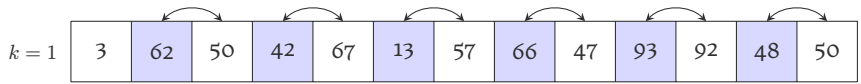

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$k = 1$	3	50	62	42	67	13	57	47	66	92	93	48	50

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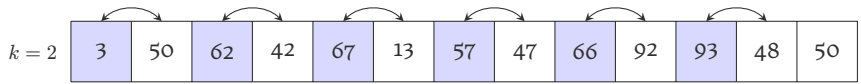

Sorting on a Mesh-Connected MultiComputer

```
function EVEN-ODD SORT(a)                                   ▷ a is an array of size n
    for k = 1, ... do
        if k is even then
            Compare (and swap if needed) every even position with its right neighbor;
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```

$k = 2$	3	50	42	62	13	67	47	57	66	92	48	93	50

Sorting on a Mesh-Connected MultiComputer

```
function EVEN-ODD SORT(a)                                   ▷ a is an array of size n
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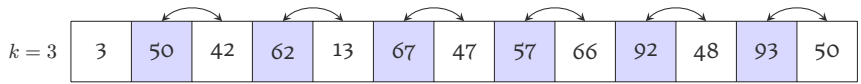

Sorting on a Mesh-Connected MultiComputer

```
function EVEN-ODD SORT(a)                                   ▷ a is an array of size n
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        if k is even then
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            Compare (and swap if needed) every odd position with its right neighbor.
```

$k = 3$	3	42	50	13	62	47	67	57	66	48	92	50	93

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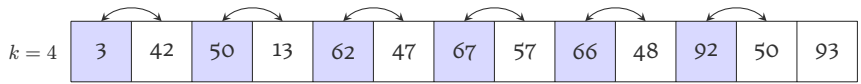

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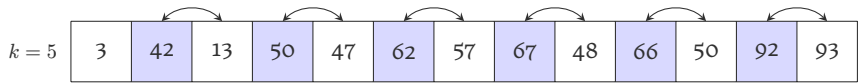
$k = 4$	3	42	13	50	47	62	57	67	48	66	50	92	93

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
    if k is even **then**
      Compare (and swap if needed) every even position with its right neighbor;
    else (k is odd)
      Compare (and swap if needed) every odd position with its right neighbor.



Sorting on a Mesh-Connected MultiComputer

```
function EVEN-ODD SORT(a)                                   ▷ a is an array of size n
    for k = 1, ... do
        if k is even then
            Compare (and swap if needed) every even position with its right neighbor;
        else (k is odd)
            Compare (and swap if needed) every odd position with its right neighbor.
```

$k = 5$	3	13	42	47	50	57	62	48	67	50	66	92	93

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
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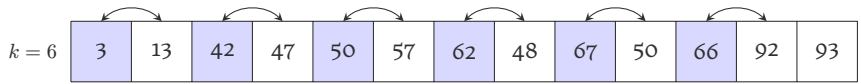

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
    if k is even **then**
      Compare (and swap if needed) every even position with its right neighbor;
    else (k is odd)
      Compare (and swap if needed) every odd position with its right neighbor.

$k = 6$	3	13	42	47	50	57	48	62	50	67	66	92	93

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
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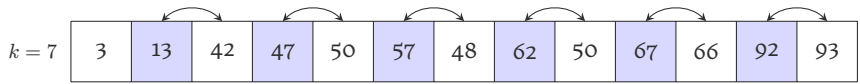

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
    if k is even **then**
      Compare (and swap if needed) every even position with its right neighbor;
    else (k is odd)
      Compare (and swap if needed) every odd position with its right neighbor.

$k = 7$	3	13	42	47	50	48	57	50	62	66	67	92	93

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
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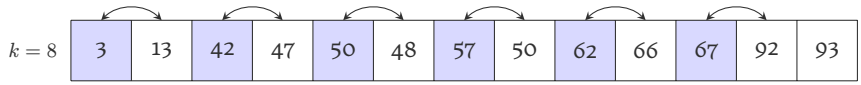


Sorting on a Mesh-Connected MultiComputer

```
function EVEN-ODD SORT(a)                                   ▷ a is an array of size n
    for k = 1, ... do
        if k is even then
            Compare (and swap if needed) every even position with its right neighbor;
        else (k is odd)
            Compare (and swap if needed) every odd position with its right neighbor.
```

$k = 8$	3	13	42	47	48	50	50	57	62	66	67	92	93

Sorting on a Mesh-Connected MultiComputer

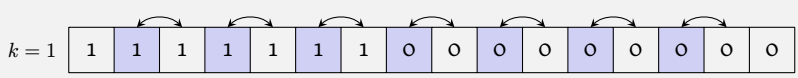
function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
    if k is even **then**
      Compare (and swap if needed) every even position with its right neighbor;
    else (k is odd)
      Compare (and swap if needed) every odd position with its right neighbor.

$k = 8$	3	13	42	47	48	50	50	57	62	66	67	92	93

Lemma

If the array a has size n, then it is sorted after the iteration $k = n$ of EVEN-ODD SORT.

Sketch of proof.



□

Sorting on a Mesh-Connected MultiComputer

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$k = 8$	3	13	42	47	48	50	50	57	62	66	67	92	93

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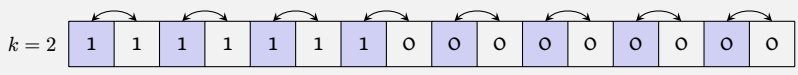

□

Sorting on a Mesh-Connected MultiComputer

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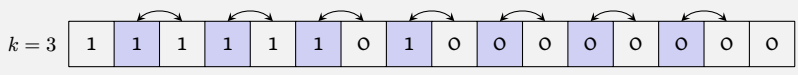

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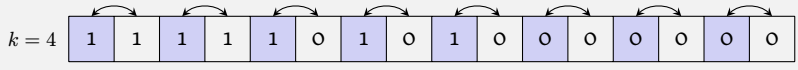

□

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
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$k = 8$	3	13	42	47	48	50	50	57	62	66	67	92	93

Lemma

If the array a has size n, then it is sorted after the iteration $k = n$ of EVEN-ODD SORT.

Sketch of proof.

$k = 5$	1	1	1	1	0	1	0	1	0	1	0	0	0	0	0	0

□

Sorting on a Mesh-Connected MultiComputer

function EVEN-ODD SORT(a) ▷ *a is an array of size n*
  for $k = 1, \ldots$ **do**
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$k = 8$	3	13	42	47	48	50	50	57	62	66	67	92	93

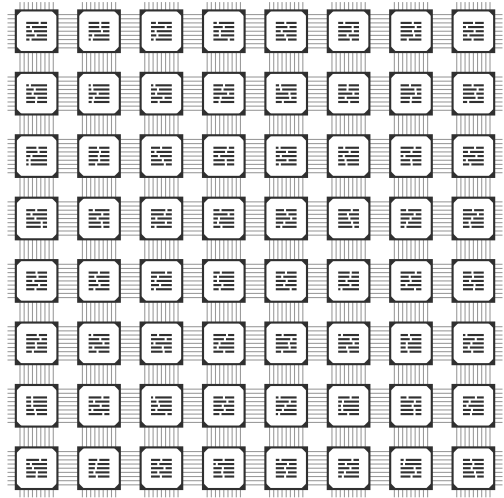
Lemma

If the array a has size n, then it is sorted after the iteration $k = n$ of EVEN-ODD SORT.

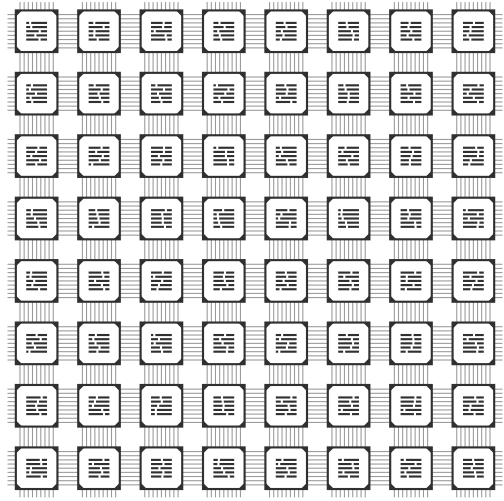
Corollary

The SORTING problem on MCMCs can be solved in time $O(n)$ on arrays of size n.

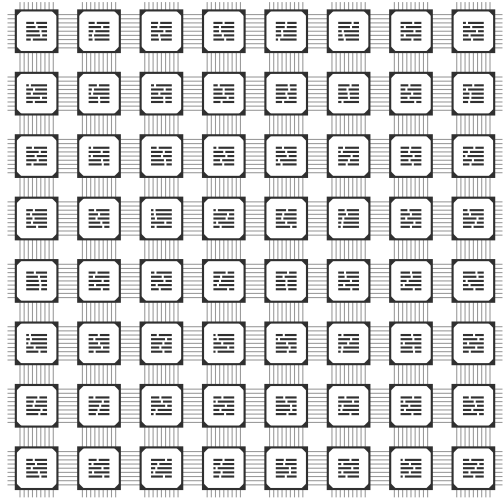
Sorting on a 2D Mesh-Connected MultiComputer



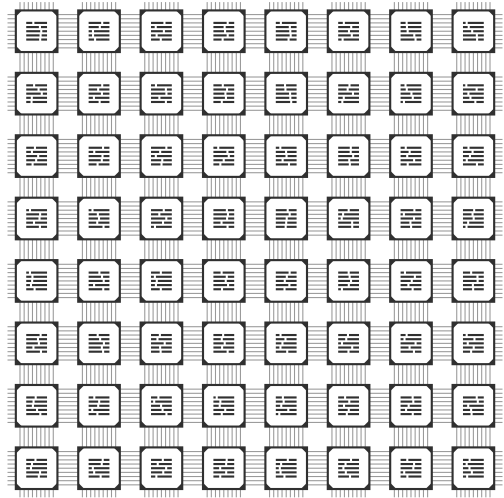
Sorting on a 2D Mesh-Connected MultiComputer



Sorting on a 2D Mesh-Connected MultiComputer



Sorting on a 2D Mesh-Connected MultiComputer



Sorting on a 2D Mesh-Connected MultiComputer

function SNAKE SORT(a) ▷ *a is an array of size $\sqrt{n} \times \sqrt{n}$*

  for $k = 1, \ldots$ **do**

    Sort rows in increasing and decreasing order alternatively;

    Sort columns in increasing order.

11	80	47	13	33	5	9	99	77	24
91	5	24	80	85	43	39	53	60	67
55	8	9	50	13	94	77	50	24	28
32	97	20	43	45	60	79	23	71	89
75	16	59	88	72	4	90	53	3	3
48	39	24	18	21	50	37	21	12	30
5	11	82	99	11	14	47	89	43	1
20	56	61	58	95	68	45	79	78	12
92	48	50	13	92	96	18	50	14	53
3	62	50	42	67	13	57	66	47	93

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    Sort columns in increasing order.

$k = 1$

11	80	47	13	33	5	9	99	77	24
91	5	24	80	85	43	39	53	60	67
55	8	9	50	13	94	77	50	24	28
32	97	20	43	45	60	79	23	71	89
75	16	59	88	72	4	90	53	3	3
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75	16	59	88	72	4	90	53	3	3
48	39	24	18	21	50	37	21	12	30
5	11	82	99	11	14	47	89	43	1
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32	97	20	43	45	60	79	23	71	89
75	16	59	88	72	4	90	53	3	3
48	39	24	18	21	50	37	21	12	30
5	11	82	99	11	14	47	89	43	1
20	56	61	58	95	68	45	79	78	12
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32	97	20	43	45	60	79	23	71	89
75	16	59	88	72	4	90	53	3	3
48	39	24	18	21	50	37	21	12	30
5	11	82	99	11	14	47	89	43	1
20	56	61	58	95	68	45	79	78	12
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$k = 1$

5	9	11	13	24	33	47	77	80	99
91	85	80	67	60	53	43	39	24	5
8	9	13	24	28	50	50	55	77	94
32	97	20	43	45	60	79	23	71	89
75	16	59	88	72	4	90	53	3	3
48	39	24	18	21	50	37	21	12	30
5	11	82	99	11	14	47	89	43	1
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$k = 1$

5	9	11	13	24	33	47	77	80	99
91	85	80	67	60	53	43	39	24	5
8	9	13	24	28	50	50	55	77	94
32	97	20	43	45	60	79	23	71	89
75	16	59	88	72	4	90	53	3	3
48	39	24	18	21	50	37	21	12	30
5	11	82	99	11	14	47	89	43	1
20	56	61	58	95	68	45	79	78	12
92	48	50	13	92	96	18	50	14	53
3	62	50	42	67	13	57	66	47	93

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  for $k = 1, \ldots$ **do**

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    Sort columns in increasing order.

$k = 1$

5	9	11	13	24	33	47	77	80	99
91	85	80	67	60	53	43	39	24	5
8	9	13	24	28	50	50	55	77	94
97	89	79	71	60	45	43	32	23	20
75	16	59	88	72	4	90	53	3	3
48	39	24	18	21	50	37	21	12	30
5	11	82	99	11	14	47	89	43	1
20	56	61	58	95	68	45	79	78	12
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3	62	50	42	67	13	57	66	47	93

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    Sort columns in increasing order.

$k = 1$

5	9	11	13	24	33	47	77	80	99
91	85	80	67	60	53	43	39	24	5
8	9	13	24	28	50	50	55	77	94
97	89	79	71	60	45	43	32	23	20
3	3	4	16	53	59	72	75	88	90
50	48	39	37	30	24	21	21	18	12
1	5	11	11	14	43	47	82	89	99
95	79	78	68	61	58	56	45	20	12
13	14	18	48	50	50	53	92	92	96
93	67	66	62	57	50	47	42	13	3

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8	9	13	24	28	50	50	55	77	94
97	89	79	71	60	45	43	32	23	20
3	3	4	16	53	59	72	75	88	90
50	48	39	37	30	24	21	21	18	12
1	5	11	11	14	43	47	82	89	99
95	79	78	68	61	58	56	45	20	12
13	14	18	48	50	50	53	92	92	96
93	67	66	62	57	50	47	42	13	3

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1	9	11	13	24	33	47	77	80	99
3	85	80	67	60	53	43	39	24	5
5	9	13	24	28	50	50	55	77	94
8	89	79	71	60	45	43	32	23	20
13	3	4	16	53	59	72	75	88	90
50	48	39	37	30	24	21	21	18	12
91	5	11	11	14	43	47	82	89	99
93	79	78	68	61	58	56	45	20	12
95	14	18	48	50	50	53	92	92	96
97	67	66	62	57	50	47	42	13	3

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  for $k = 1, \ldots$ **do**

    Sort rows in increasing and decreasing order alternatively;

    Sort columns in increasing order.

$k = 1$

1	3	4	11	14	24	21	21	13	3
3	5	11	13	24	33	43	32	18	5
5	9	11	16	28	43	43	39	20	12
8	9	13	24	30	45	47	42	23	12
13	14	18	37	50	50	47	45	24	20
50	48	39	48	53	50	47	55	77	90
91	67	66	62	57	50	50	75	80	94
93	79	78	67	60	53	53	77	88	96
95	85	79	68	60	58	56	82	89	99
97	89	80	71	61	59	72	92	92	99

Sorting on a 2D Mesh-Connected MultiComputer

function SNAKE SORT(a) ▷ *a is an array of size $\sqrt{n} \times \sqrt{n}$*

  for $k = 1, \ldots$ **do**

    Sort rows in increasing and decreasing order alternatively;

    Sort columns in increasing order.

$k = 2$

1	3	3	4	11	13	14	21	21	24
43	33	32	24	18	13	11	5	5	3
5	9	11	12	16	20	28	39	43	43
47	45	42	30	24	23	13	12	9	8
13	14	18	20	24	37	45	47	50	50
90	77	55	53	50	50	48	48	47	39
50	50	57	62	66	67	75	80	91	94
96	93	88	79	78	77	67	60	53	53
56	58	60	68	79	82	85	89	95	99
99	97	92	92	89	80	72	71	61	59

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    Sort columns in increasing order.

$k = 2$

1	3	3	4	11	13	11	5	5	3
5	9	11	12	16	13	13	12	9	8
13	14	18	20	18	20	14	21	21	24
43	33	32	24	24	23	28	39	43	39
47	45	42	30	24	37	45	47	47	43
50	50	55	53	50	50	48	48	50	50
56	58	57	62	66	67	67	60	53	53
90	77	60	68	78	77	72	71	61	59
96	93	88	79	79	80	75	80	91	94
99	97	92	92	89	82	85	89	95	99

Sorting on a 2D Mesh-Connected MultiComputer

function SNAKE SORT(a) ▷ *a is an array of size $\sqrt{n} \times \sqrt{n}$*

  for $k = 1, \ldots$ **do**

    Sort rows in increasing and decreasing order alternatively;

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$k = 3$

1	3	3	3	4	5	5	11	11	13
16	13	13	12	12	11	9	9	8	5
13	14	14	18	18	20	20	21	21	24
43	43	39	39	33	32	28	24	24	23
24	30	37	42	43	45	45	47	47	47
55	53	50	50	50	50	50	50	48	48
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Sketch of proof.

0	0	0	0	1	1	1	1	1	1
1	1	0	0	0	0	0	0	0	0

□

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1	1	0	0	1	1	1	1	1	1

0	0	0	0	1	1	1	0	0	0
1	1	1	1	1	1	1	1	1	1

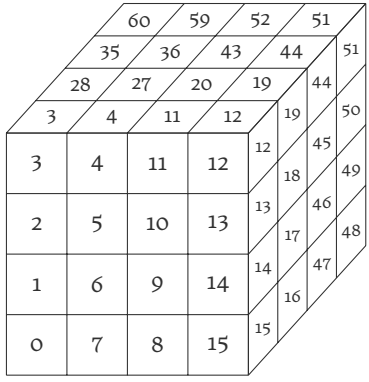
The number of "mixed" rows is halved after each iteration, hence $\log_2(\sqrt{n})$ iterations. □

Corollary

The SORTING problem on MCMCs can be solved in time $O(\sqrt{n} \cdot \log n)$ on arrays of size $\sqrt{n} \times \sqrt{n}$.

Sorting on a d-dimensional Mesh-Connected MultiComputer

Snake ordering generalizes to higher dimensions:



Theorem ([Corbett & Scherson, 1992])

The SORTING problem on MCMCs can be solved in time $O(\sqrt[d]{n} \cdot \log n)$ on arrays of size $\underbrace{\sqrt[d]{n} \times \cdots \times \sqrt[d]{n}}_{d \text{ times}}$.

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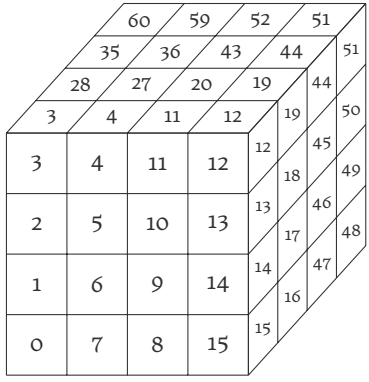

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Fast computing with tilings

The RAM model

Definition

A *Random Access Machine* is composed of:

- A finite sequence of *instructions* called a *program* (and an *instruction pointer* addressing the instruction currently being executed);
- A finite set of variables $\texttt{var}_1, \dots, \texttt{var}_k$, each containing an integer;
- A *memory array* M composed of infinitely many *memory cells* $(M_i)_{i \in \mathbb{N}}$, each containing an integer.

Instructions are of two types:

- Arithmetic instructions on variables, *e.g.* $\texttt{var}_i \leftarrow \texttt{var}_j + \texttt{var}_k$;
- Memory instructions, either reading $\texttt{var}_i \leftarrow M_j$ or writing $M_i \leftarrow \texttt{var}_j$.

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Lemma

The RAM model is Turing-complete.

Simulating RAM computations with Mesh-Connected MultiComputers

Fix $n \in \mathbb{N}$. We simulate n computations steps of a RAM program p in an MCMC of size n:

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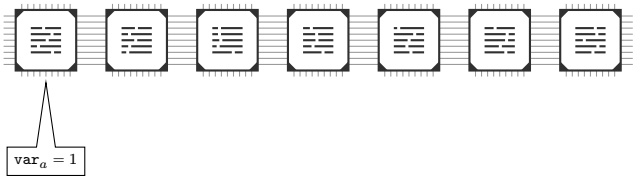

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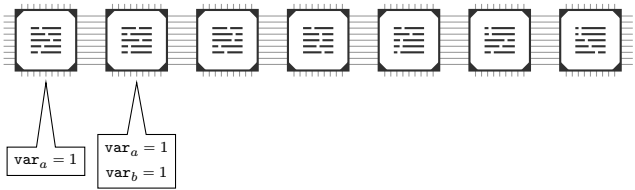

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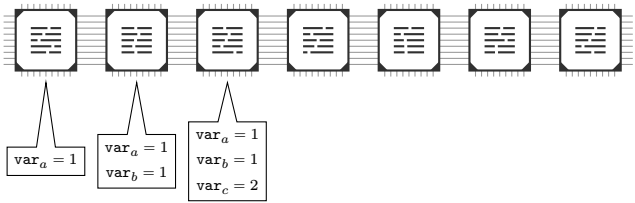

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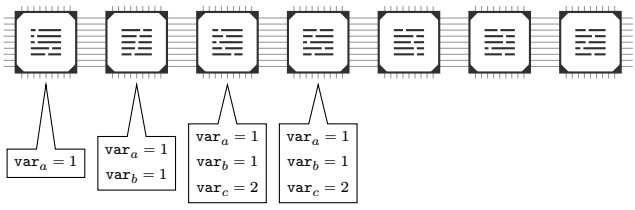

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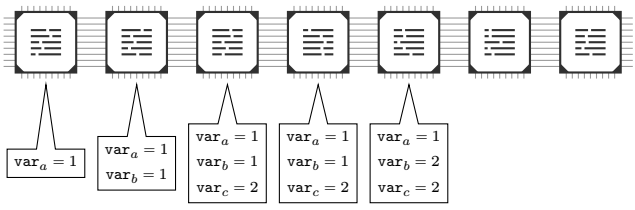

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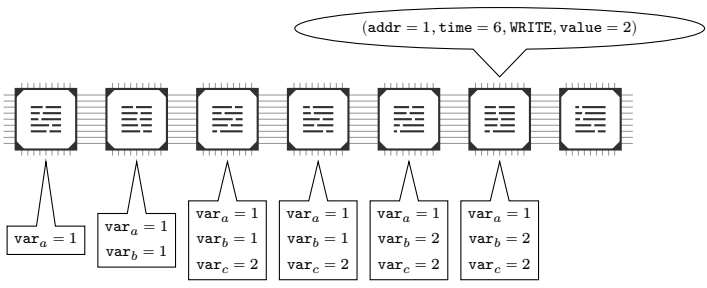

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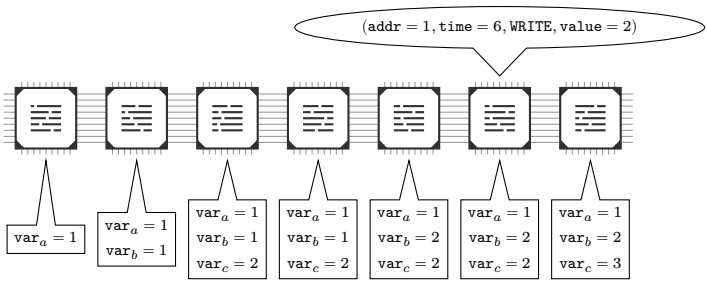

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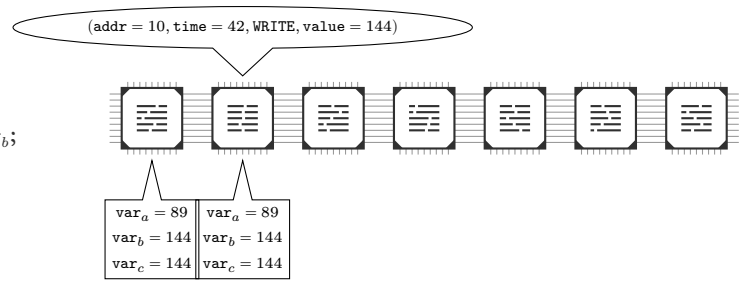

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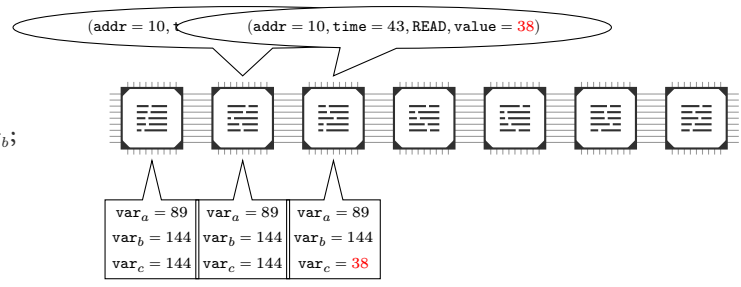

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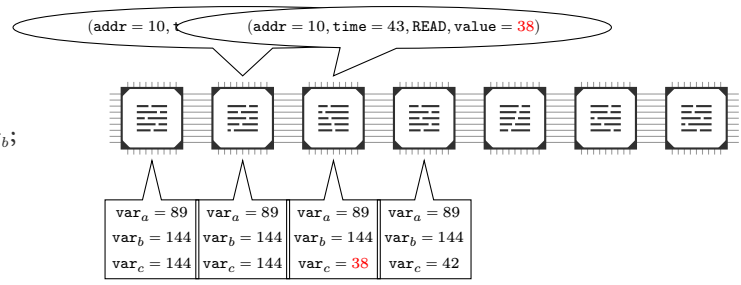

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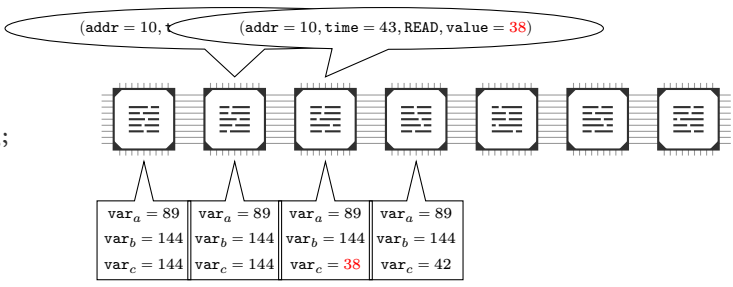

At the end of the procedure, we have a list of records of the form:

- Writing records: $(\texttt{addr}, \texttt{time}, \texttt{WRITE}, \texttt{value})$;
- Reading records: $(\texttt{addr}, \texttt{time}, \texttt{READ}, \texttt{value})$;

By sorting this list in lexicographic order, we can check in time $O(1)$ the consistency of the memory guesses!

Simulating RAM computations with Mesh-Connected MultiComputers

Fix $n \in \mathbb{N}$. We simulate n computations steps of a RAM program p in an MCMC of size n:

procedure

- **for** $i = 1, \dots, n$ **do** ▷ *Parallel time* $O(1)$.
  - The i^{th} processor performs the i^{th} step of the program p.
  - If this computation step involves a *reading a memory cell* (resp. writing), the processor stores a *memory record* $(\texttt{addr}, \texttt{time}, \texttt{READ}, \texttt{value})$ (resp. $(\texttt{addr}, \texttt{time}, \texttt{WRITE}, \texttt{value})$).
- Sort the memory records lexicographically. ▷ *Parallel time* $O(\sqrt[4]{n})$.
- Check the consistency of the memory guesses. ▷ *Parallel time* $O(1)$.

Simulating RAM computations with Mesh-Connected MultiComputers

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Theorem

A cubic d-dimensional Mesh-Connected MultiComputer of edge length n can simulate n^d steps of RAM computations in time $O(n)$.

Conclusion

Take-home message

A 3-dimensional cube of size $n \times n \times n$ can embed $O(n^2)$ steps of arbitrary RAM computations.

Actually, technical details:

- Integers appearing in the RAM computations should be "small enough" to fit in the memory of a single MCMC processor (*e.g.* $O(\log n)$ bits): $\implies$ **word-RAM model**;
- When implementing with Wang tiles, the colors should at least contain the state of an MCMC processor (*e.g.* $O(\log n)$ bits): **non-constant!**

$\implies$ Mesh-Connected MultiComputers are a natural computation model for the "fixpoint construction" (Durand, Romaschenko and Shen).

Save the date!

What can we do with this? $\implies$ Prove the soficity of many multidimensional subshifts!

(June, 24th in Caen)


That's all Folks!