Information, communication and complexity of sofic tilings

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Tiling spaces

Definition

A *tiling space* on $\mathbb{Z}^d$ is a set of colorings $\mathbb{Z}^d \to \mathcal{A}$ (“configurations”) that:

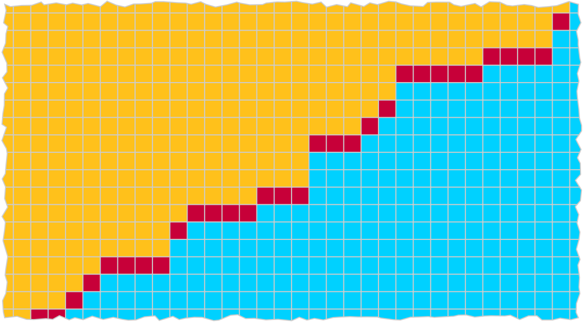
i) is closed and translation-invariant in the Cantor space $\mathcal{A}^{\mathbb{Z}^d}$;

ii) is defined by forbidden patterns $\mathcal{F}$.

$$X_{\mathcal{F}} = \left\{x \in \mathcal{A}^{\mathbb{Z}^d} : \forall w \in \mathcal{F},\, w \text{ does not appear in } x\right\}.$$

Example:

$\mathcal{F} = \{$  $\}$



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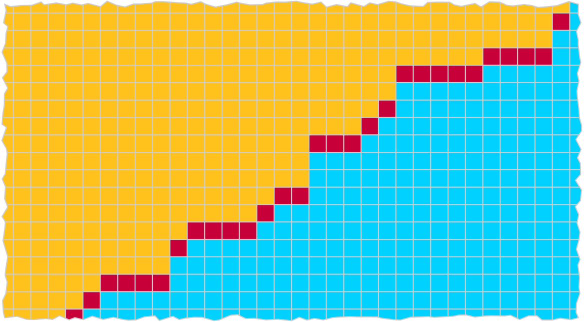

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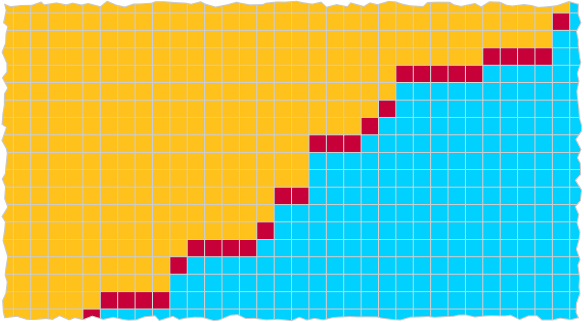

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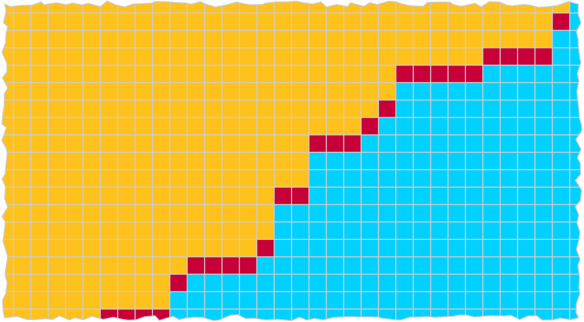

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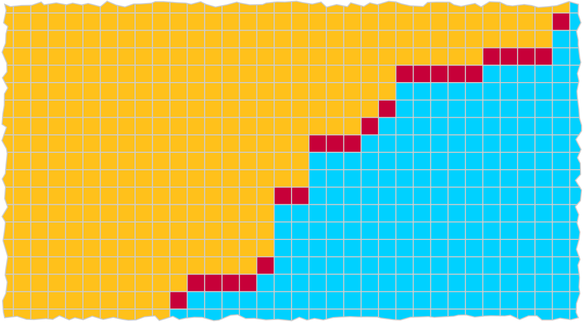

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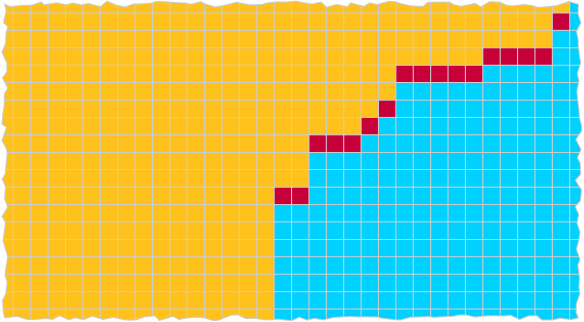

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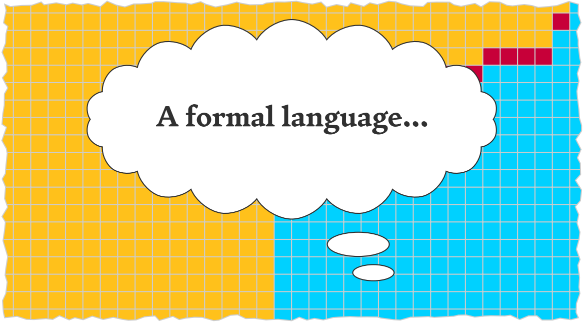

Classifying tiling spaces by computational expressive power

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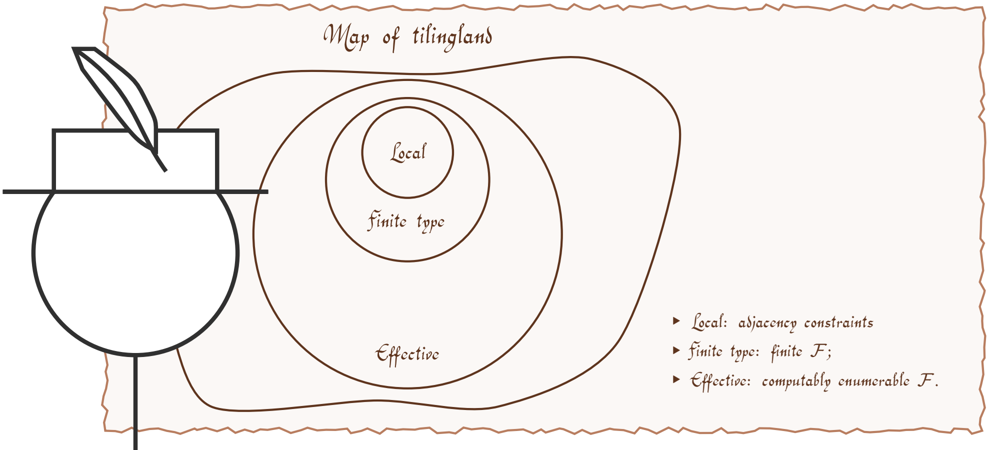

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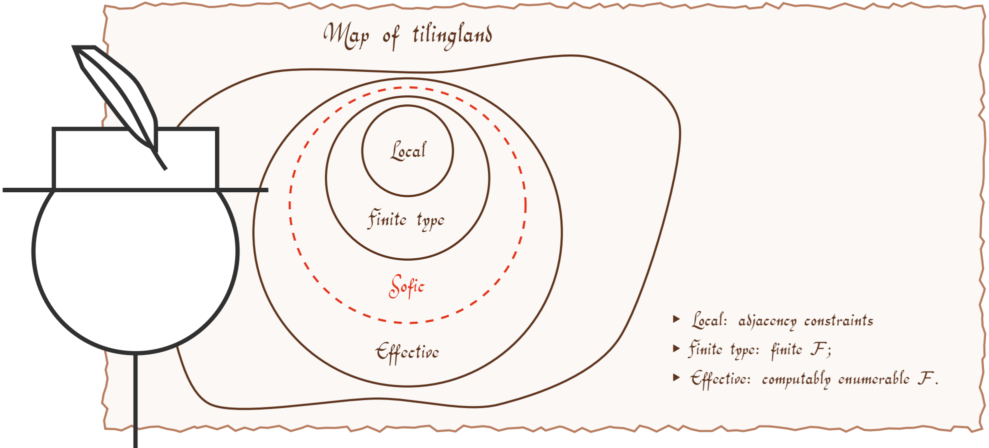

- Local: adjacency constraints
- Finite type: finite $\mathcal{F}$;
- Effective: computably enumerable $\mathcal{F}$.

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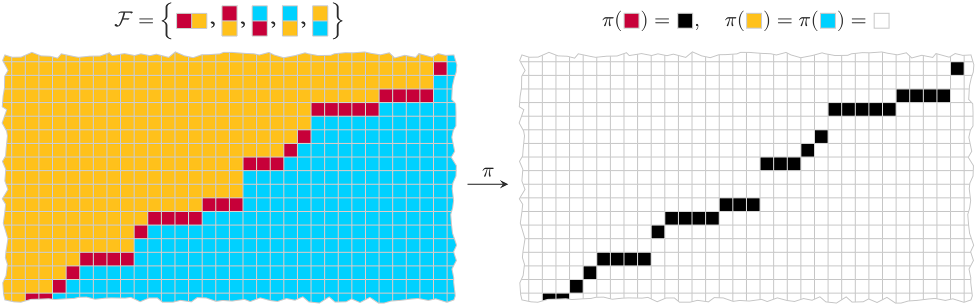
Definition (Sofic space, ≃ 1973)

A tiling space is *sofic* if it can be defined as the cell-by-cell projection of a local space.



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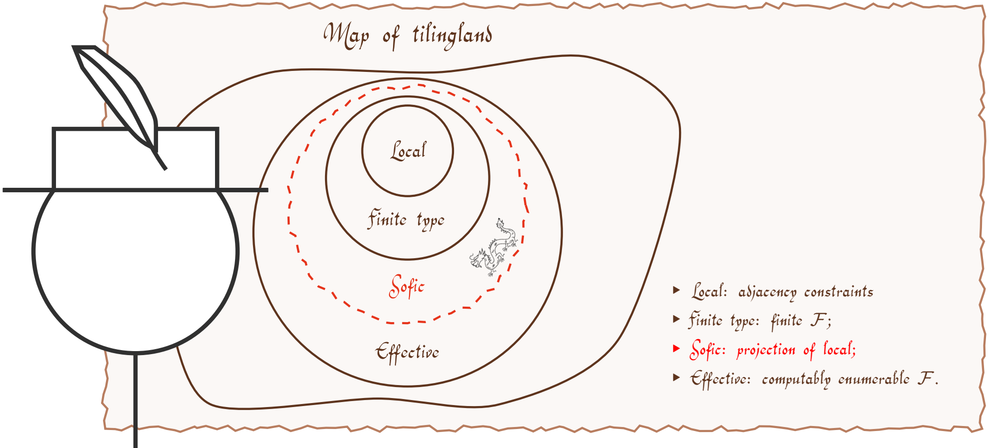

Tiling spaces on $\mathbb{Z}$

Tiling space $X \subseteq \mathcal{A}^{\mathbb{Z}}$	**Language** $L \subseteq \mathcal{A}^*$
Local/finite type	Local
Sofic	Rational/regular
Effective	Comp. co-enumerable

Example

The sunny-side-up $X_{\blacksquare} \subseteq \{\square, \blacksquare\}^{\mathbb{Z}}$ is sofic.

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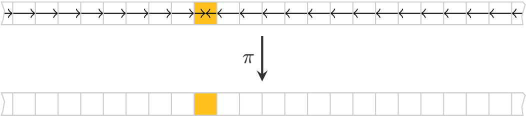

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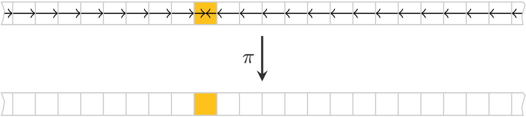

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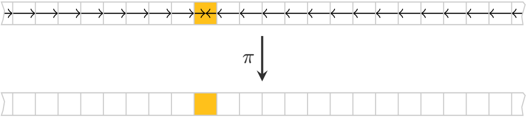

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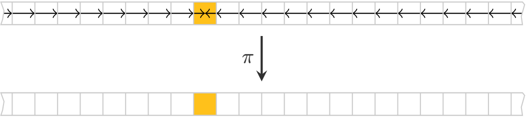

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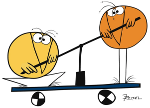

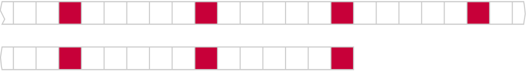

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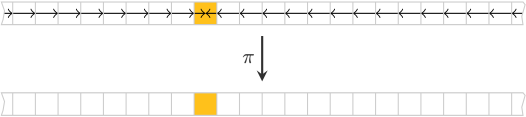

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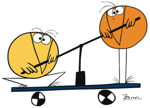

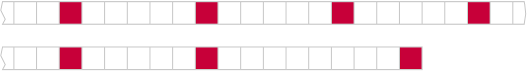

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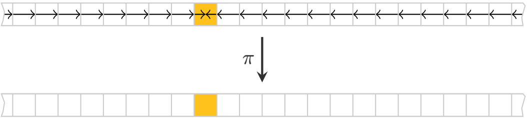

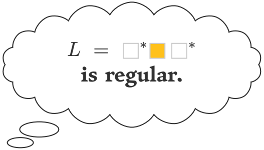

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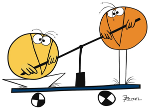

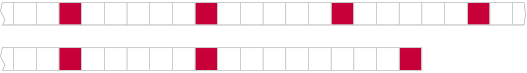

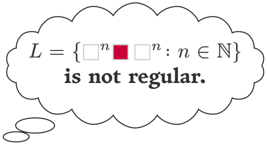

Soficity of tiling spaces on $\mathbb{Z}^2$: more expressive

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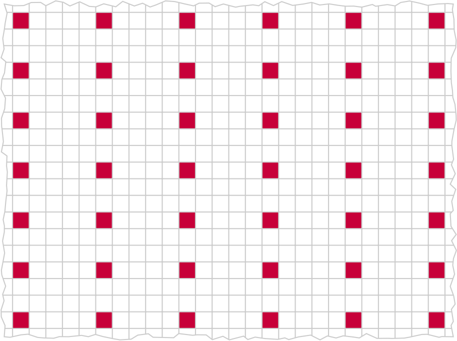
On $\mathbb{Z}^2$, the space of all periods X_p is sofic:



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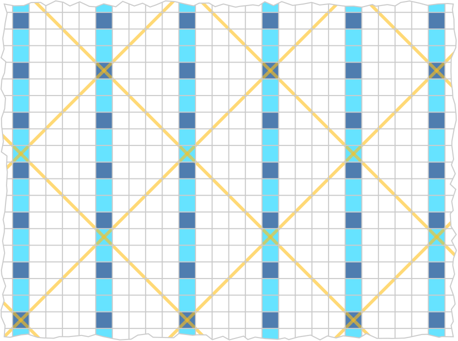

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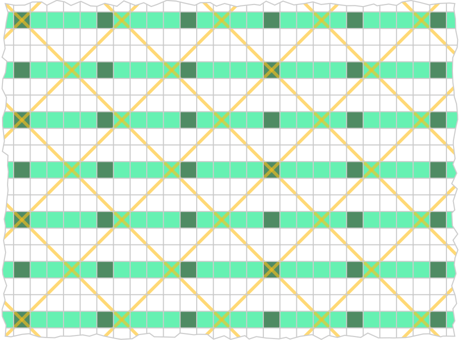

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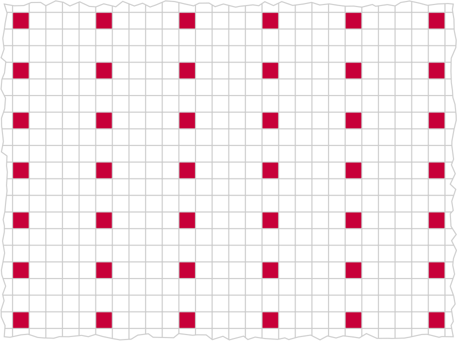

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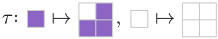
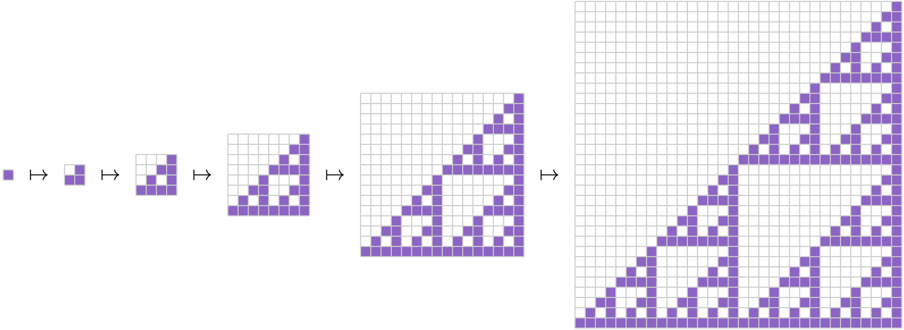
On $\mathbb{Z}^2$, substitutive tiling spaces are sofic:





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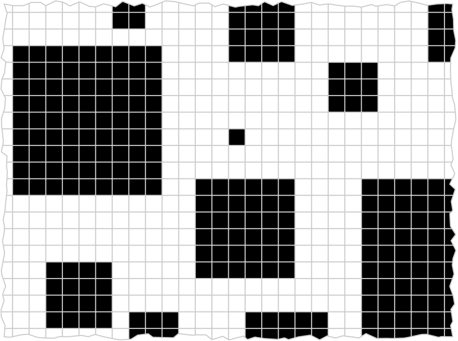
On $\mathbb{Z}^2$, the "seas of squares" space X_S is sofic:



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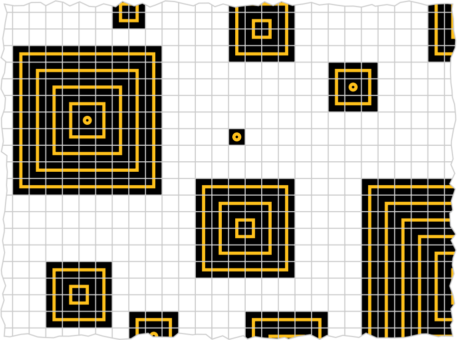

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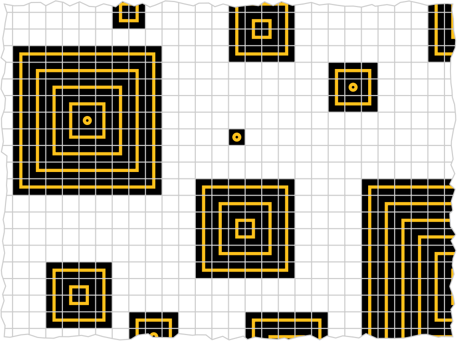
On $\mathbb{Z}^2$, the "seas of squares" space X_S is sofic, even with side lengths restricted to any Π^0_1-computable set $S \subseteq \mathbb{N}$:



[Westrick, 2017]

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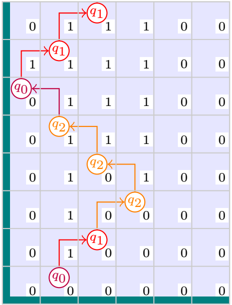
Fact

On $\mathbb{Z}^2$, we can embed computations by drawing space-time diagrams of Turing machines.



Corollary

Given $\mathcal{F}$, deciding whether there exists a valid configuration in $X_{\mathcal{F}}$ is Π_1^0-complete [Berger, 1964].
Given $\mathcal{F}$ and a pattern w, deciding whether w is a valid pattern in $X_{\mathcal{F}}$ is Π_1^0-complete...

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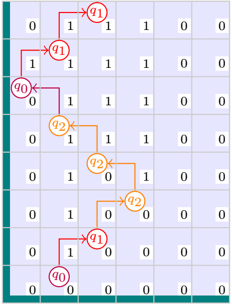

Corollary

"Sofic $\mathbb{Z}^2$ tiling spaces can be computationally complex: everything about them is undecidable."

Soficity of tiling spaces on $\mathbb{Z}^2$: more expressive

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On $\mathbb{Z}^2$, we can embed computations by drawing space-time diagrams of Turing machines.

Theorem (Higman-like simulations, $\simeq$ 2009)

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be an effective tiling space. Then its periodic extension $X^{\uparrow} \subseteq \mathcal{A}^{\mathbb{Z}^{d+1}}$ is a sofic space.

$$X^{\uparrow} = \{x \in \mathcal{A}^{\mathbb{Z}^{d+1}} : \exists x' \in X, \forall n \in \mathbb{Z},\ x|_{\mathbb{Z}^d \times \{n\}} = x'\}$$

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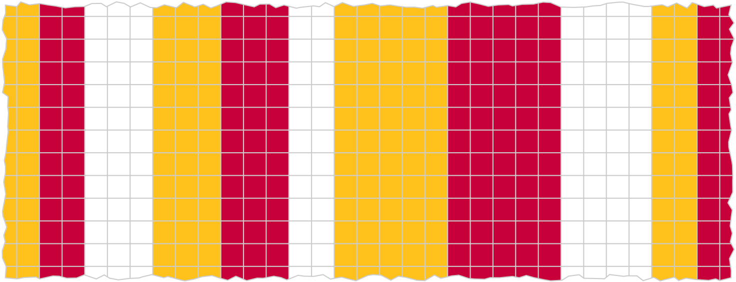
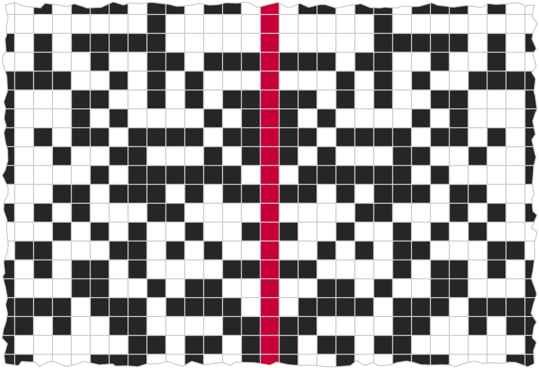


Soficity of tiling spaces on $\mathbb{Z}^2$: expressiveness limits

Example

The mirror tiling space is **not** sofic on $\mathbb{Z}^d$ for any $d \in \mathbb{N}$:



(folklore)

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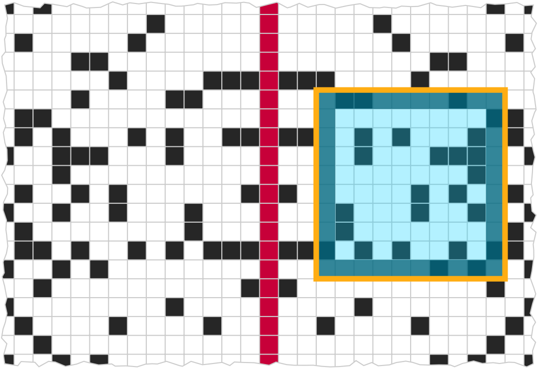

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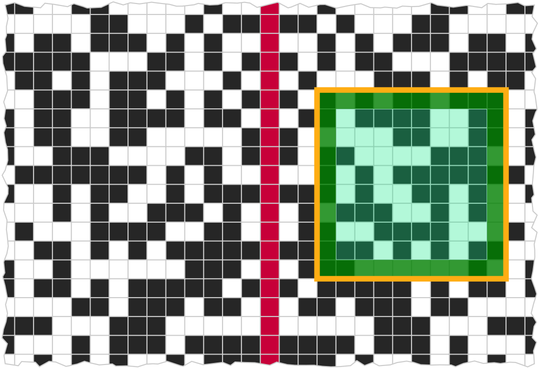
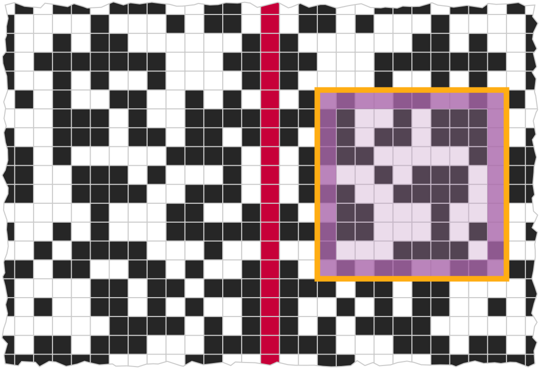


(folklore)

Soficity of tiling spaces on $\mathbb{Z}^2$: expressiveness limits

Example

The mirror tiling space is **not** sofic on $\mathbb{Z}^d$ for any $d \in \mathbb{N}$:



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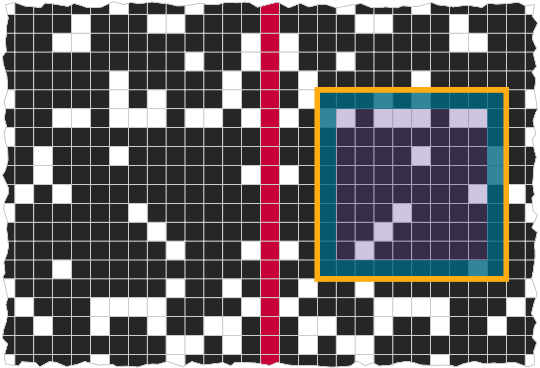

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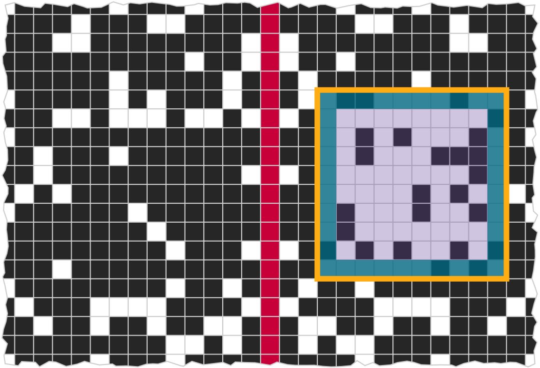

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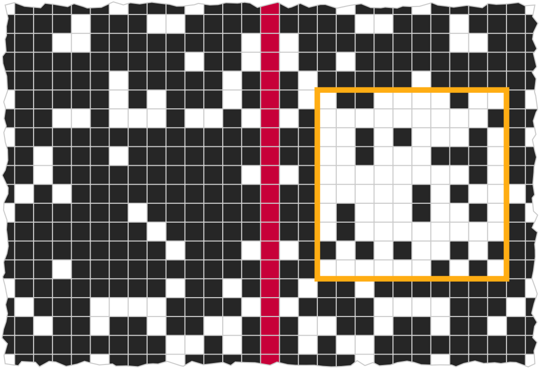

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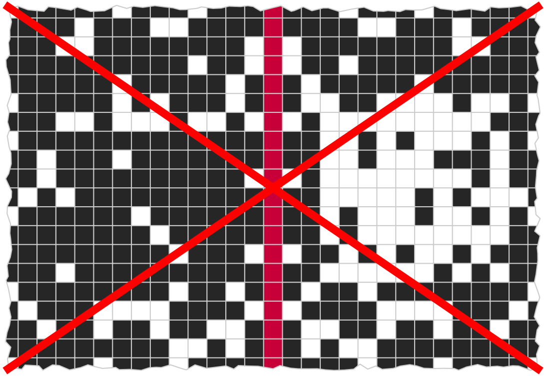

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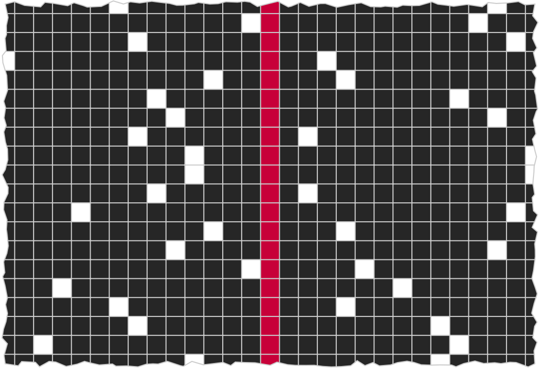
And most of the time...

Question

Is the tiling space X_{NEQ} sofic on $\mathbb{Z}^2$?



We don't know!

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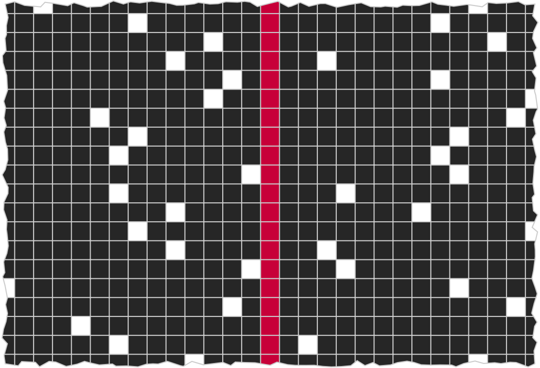


We don't know!

Communication complexity

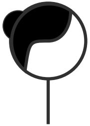
Communication complexity

Definition

Communication complexity quantifies the communication required to solve a distributed problem.

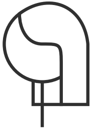
In a **communication problem**, we compute a problem $P\colon \{0,1\}^* \times \{0,1\}^* \to \{\top, \bot\}$.

Alice



$w_A \in \{0,1\}^*$

Bob



$w_B \in \{0,1\}^*$

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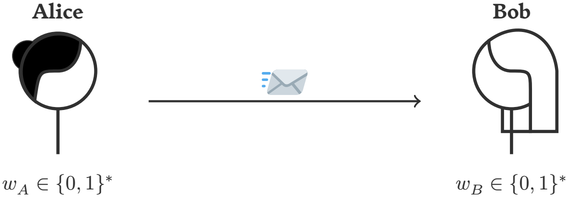

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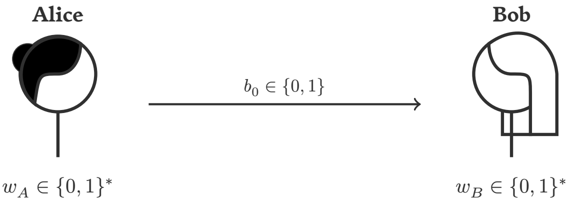

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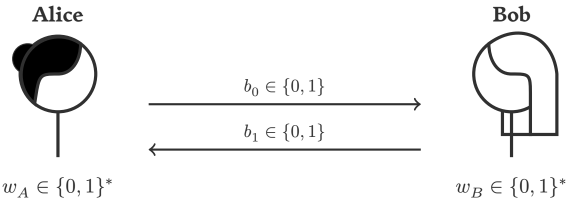

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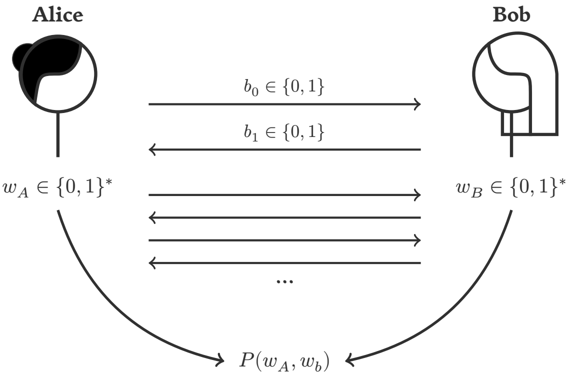


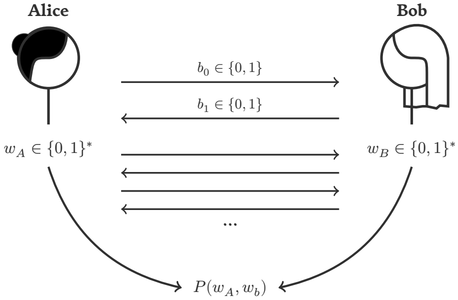
Communication complexity

Definition

For a problem P: $\{0,1\}^* \times \{0,1\}^* \to \{\top, \bot\}$, denote:

- $\mathcal{C}(P)$ the size[1] of a minimal protocol solving P;
- $\mathcal{NC}(P)$ the size of a minimal *non-deterministic*[2] protocol solving P.

[1] Where the *size* of a protocol is the length of its maximal transcript.
[2] If $P(w_A, w_B) = \top$, there must *exist* an accepting communication. Otherwise, all communications must reject.



Example

The problem EQ(n): for two words $w_A, w_B \in \{0,1\}^n$, do we have $w_A = w_B$?

$$\mathcal{C}(\text{EQ}(n)) = \Theta(n);$$
$$\mathcal{NC}(\text{EQ}(n)) = \Theta(n).$$

Example

The problem NEQ(n): for two words $w_A, w_B \in \{0,1\}^n$, do we have $w_A \neq w_B$?

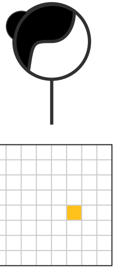
$$\mathcal{C}(\text{NEQ}(n)) = \Theta(n) (= \mathcal{C}(\text{EQ}(n)));$$
$$\mathcal{NC}(\text{NEQ}(n)) = \Theta(\log n).$$

Soficity and communication complexity

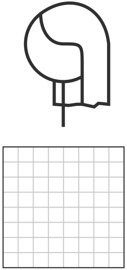
Lemma

In sofic spaces, pattern gluing of size n has *non-deterministic* communication complexity $O(n^{d-1})$.

Alice



Bob



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Protocol of complexity $O(n^{d-1})$:

1. Guess a pattern cover;
2. Communicate the border;
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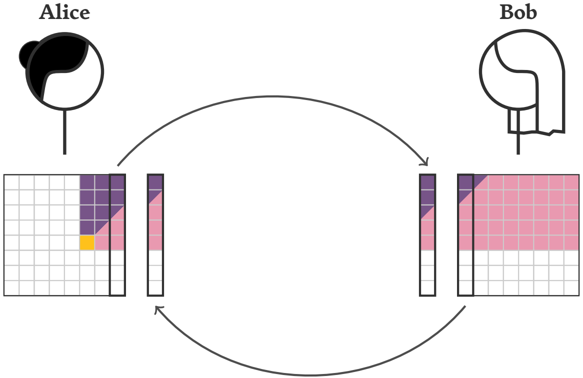

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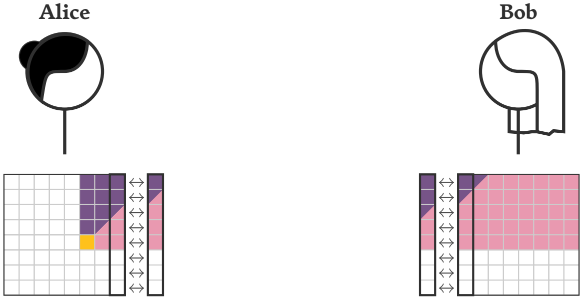

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1. On $\mathbb{Z}$, this is an equivalence:

Proposition ( )

A language $L \subseteq \mathcal{A}^*$ is iff $\mathcal{NC}(L) = O(1)$.[1]

[1] The problem L is the function $w_A, w_B \in \{0,1\}^* \times \{0,1\}^* \mapsto w_A \cdot w_B \stackrel{?}{\in} L$

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Proposition ("Myhill-Nerode" revisited)

A language $L \subseteq \mathcal{A}^*$ is regular iff $\mathcal{NC}(L) = O(1)$.[1]

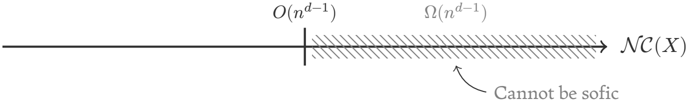
[1] The problem L is the function $w_A, w_B \in \{0,1\}^* \times \{0,1\}^* \mapsto w_A \cdot w_B \overset{?}{\in} L$

Soficity and communication complexity

Lemma

In sofic spaces, pattern gluing of size n has *non-deterministic* communication complexity $O(n^{d-1})$.

1. On $\mathbb{Z}$, this is an equivalence;
2. What about on $\mathbb{Z}^d$ for $d \geq 2$?



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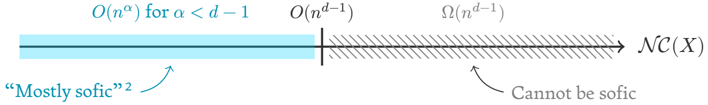

[2] Unless you pick a computationally intensive degenerate example.

Lemma (Approximation of [C., 2025])

If $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ has "*time efficient*" and "*recursive*" non-deterministic communication complexity $O(n^\alpha)$ for some $\alpha < d-1$, then X is sofic.

Soficity and communication complexity

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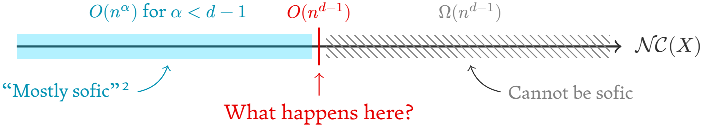
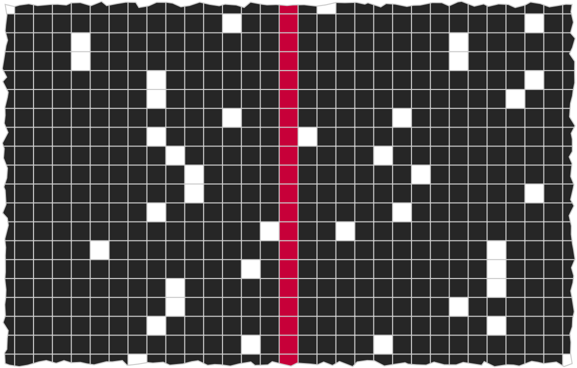


A toy example: the NEQ problem

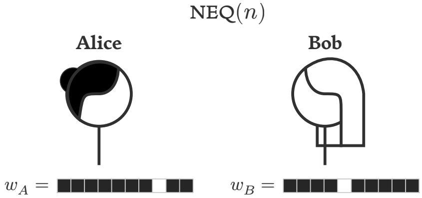
The NEQ problem



The NEQ problem

Definition (NEQ(n))

Define the communication problem NEQ$(n) = \{w_A, w_B \in \{0,1\}^n : |w_A|_1 = |w_B|_1 = 1$ and $w_A \neq \overline{w_B}\}$.



Proposition

$\mathcal{NC}(\text{NEQ}(n)) = O(\log\log n)$...

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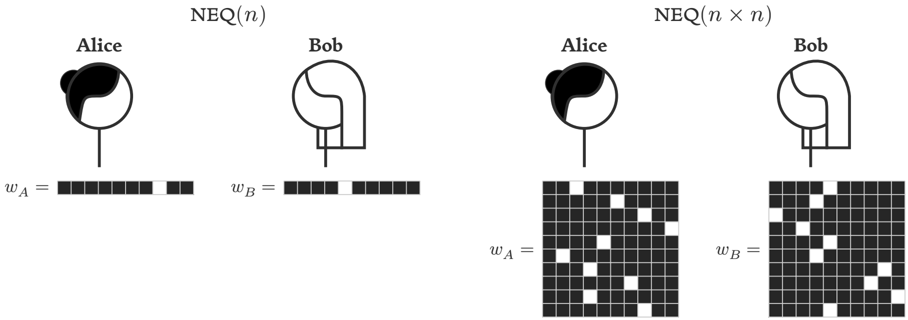

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$\mathcal{NC}(\text{NEQ}(n)) = O(\log\log n)$...

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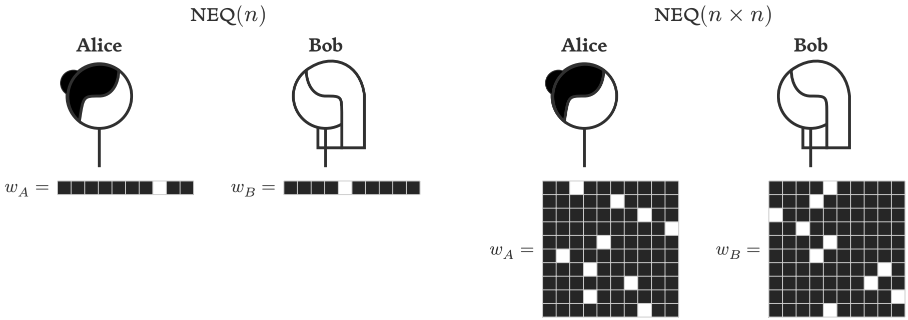

Proposition

$\mathcal{NC}(\text{NEQ}(n)) = O(\log\log n)$... but $\mathcal{NC}(\text{NEQ}(n \times n)) = O(n)$.

Question: Is the tiling space X_{NEQ} sofic on $\mathbb{Z}^2$?

A protocol for NEQ$(n \times n)$

Part 1: assuming shared randomness

$r = 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ \cdots$

Alice **Bob**

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1. Draw a random $n \times n$ matrix M;
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$r =$ 0 1 1 0 1 0 1 1 0 1 1 1 0 ⋯

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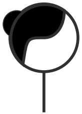

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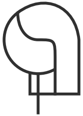

$$M = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 0&1&0&0&0&1&1&0&1&1\\ \hline 1&1&1&0&0&1&0&1&0&0\\ \hline 1&0&0&0&1&1&1&1&0&1\\ \hline 0&0&1&0&1&0&1&1&1&1\\ \hline 1&0&1&0&1&1&1&0&1&0\\ \hline 0&0&0&1&1&0&1&0&0&0\\ \hline 1&0&0&1&1&0&0&0&0&1\\ \hline 1&1&0&1&0&1&1&0&1&0\\ \hline 0&1&0&1&1&1&0&1&1&0\\ \hline 1&1&0&1&0&1&0&1&1&1\\ \hline\end{array}$$

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Part 1: assuming shared randomness

1. Draw a random $n \times n$ matrix M;
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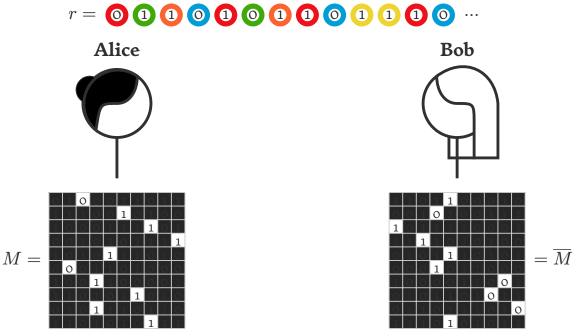

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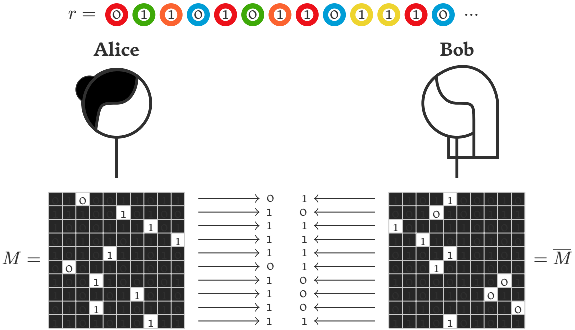

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Part 2: removing shared randomness

1. Communicate a non-deterministic seed $s \in \{0,1\}^*$;
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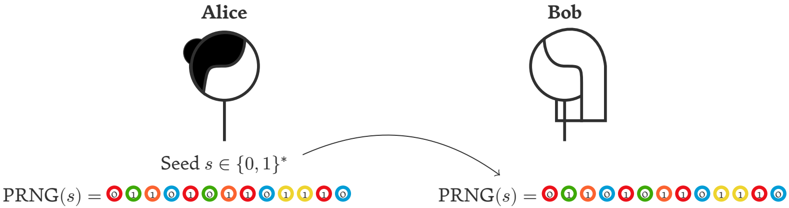

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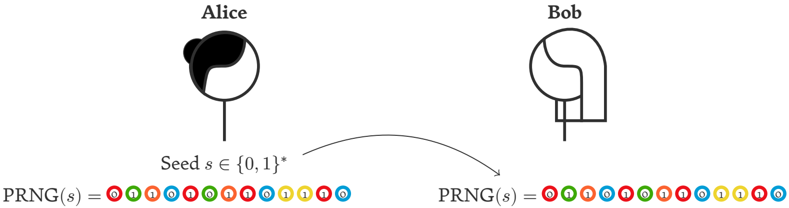

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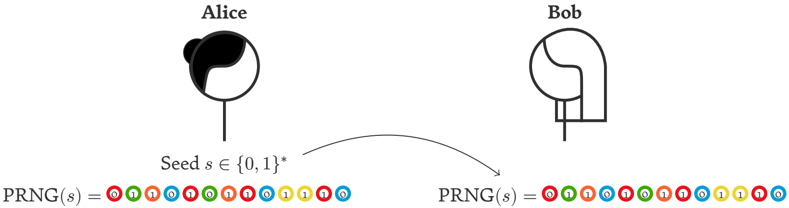
3. Communicate $(a_i)_{1\leq i\leq n}$ and $(b_i)_{i\leq i\leq n}$, and check that all $a_i \neq b_i$. Communication $O(n)$ bits



Part 2: removing shared randomness

1. Communicate a non-deterministic seed $s \in \{0,1\}^*$; Communication $O(\log n)$ bits
2. `GOTO` [Part 1] using a pseudo-random generator.

Linear Feedback Shift Registers

Definition (LFSR)

Two words $(f, s) \in \{0,1\}^m \times \{0,1\}^m$ generate a *shift register sequence* $(r_k)_{k \in \mathbb{N}}$ defined by:

$$r_k = \begin{cases} s_k & \text{if k < m} \\ \langle f, r|_{k-m,\ldots,k-1} \rangle & \text{otherwise.} \end{cases}$$

Example

For $f = 0\,0\,0\,0\,1\,1\,0\,1\,1\,1$ and $s = 1\,1\,0\,1\,0\,0\,1\,1\,0\,0$:

$$r = 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0$$

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$$r = \begin{matrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & \nearrow \end{matrix}$$

Linear Feedback Shift Registers

Definition (LFSR)

Two words $(f, s) \in \{0,1\}^m \times \{0,1\}^m$ generate a *shift register sequence* $(r_k)_{k\in\mathbb{N}}$ defined by:

$$r_k = \begin{cases} s_k & \text{if k < m} \\ \langle f, r|_{k-m,\dots,k-1}\rangle & \text{otherwise.} \end{cases}$$

Example

For $f = 0\,0\,0\,0\,1\,1\,0\,1\,1\,1$ and $s = 1\,1\,0\,1\,0\,0\,1\,1\,0\,0$:

$$\begin{array}{llllllllllll} r = & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \nearrow \end{array}$$

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Example

For $f = 0000110111$ and $s = 1101001100$:

$$r = 1\;1\;0\;1\;0\;0\;1\;1\;0\;0\;1\;0\;1$$

$$0\;0\;0\;0\;1\;1\;0\;1\;1\;1$$

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$$0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1 \nearrow$$

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$$r = 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0$$

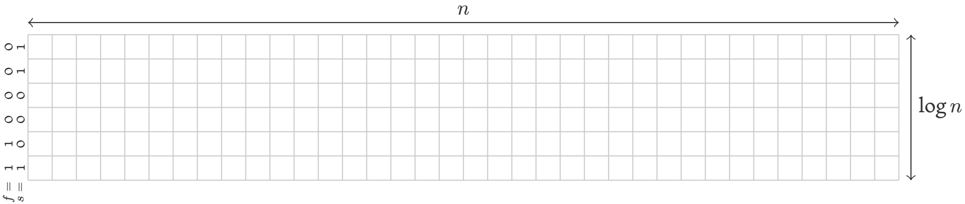
0 0 0 0 1 1 0 1 1 1

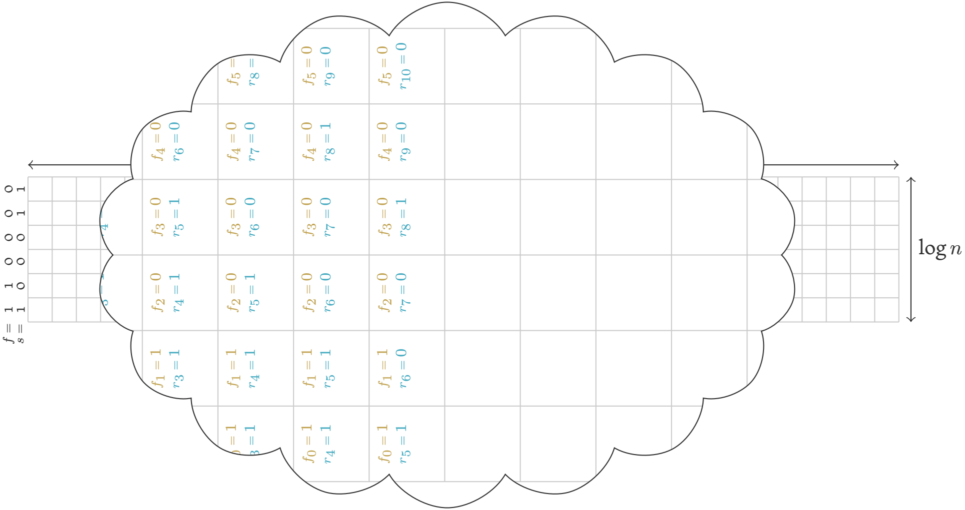
Lemma

For every $m \in \mathbb{N}$ and indices $0 \leq i_1 < \cdots < i_m < 2^m$, the set

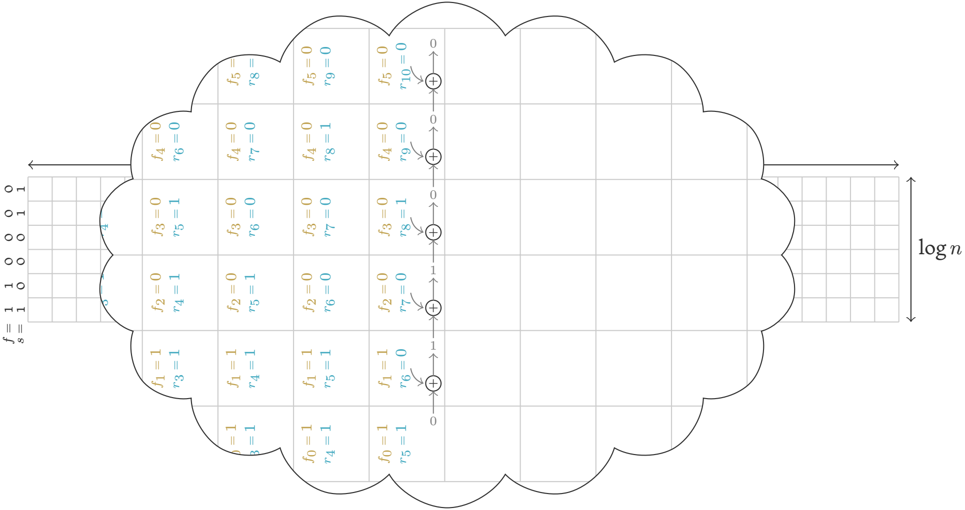
$$\{r|_{i_1,\dots,i_m} \in \{0,1\}^m : r \in \{0,1\}^{\mathbb{N}} \text{ generated by an LFSR of length } O(m)\}$$

entirely covers $\{0,1\}^m$.

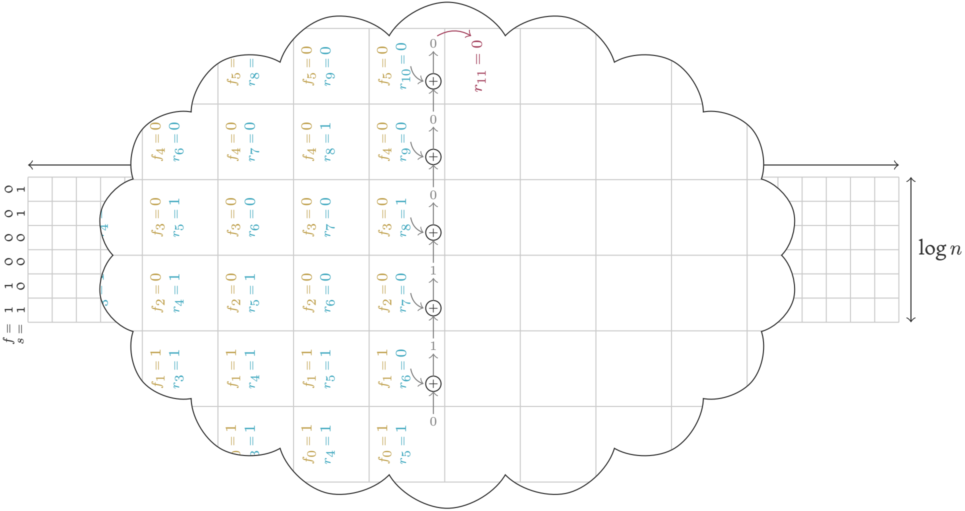

n
log n
f = 1 1 0 0 0 0
s = 1 0 0 0 1 1



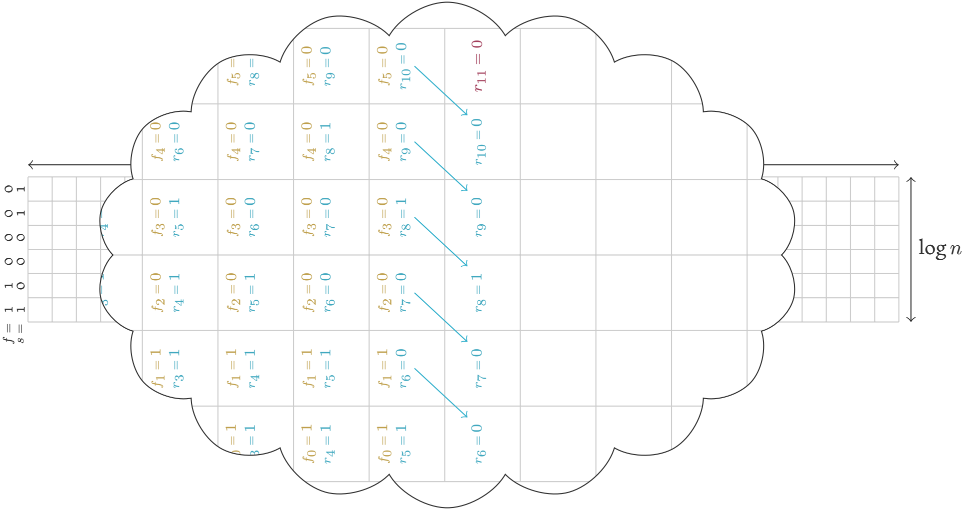
f = 1 1 0 0 0 0
s = 1 0 0 0 1 1
$f_4 = 0$ $r_6 = 0$
$f_3 = 0$ $r_5 = 1$
$f_2 = 0$ $r_4 = 1$
$f_1 = 1$ $r_3 = 1$
$f_4 = 0$ $r_7 = 0$
$f_3 = 0$ $r_6 = 0$
$f_2 = 0$ $r_5 = 1$
$f_1 = 1$ $r_4 = 1$
$f_5 = 0$ $r_9 = 0$
$f_4 = 0$ $r_8 = 1$
$f_3 = 0$ $r_7 = 0$
$f_2 = 0$ $r_6 = 0$
$f_1 = 1$ $r_5 = 1$
$f_0 = 1$ $r_4 = 1$
$f_5 = 0$ $r_{10} = 0$
$f_4 = 0$ $r_9 = 0$
$f_3 = 0$ $r_8 = 1$
$f_2 = 0$ $r_7 = 0$
$f_1 = 1$ $r_6 = 0$
$f_0 = 1$ $r_5 = 1$
$\log n$



$f =$ 1 1 0 0 0 0
$s =$ 1 0 0 0 1 1
$f_4 = 0$ $r_6 = 0$
$f_3 = 0$ $r_5 = 1$
$f_2 = 0$ $r_4 = 1$
$f_1 = 1$ $r_3 = 1$
$f_4 = 0$ $r_7 = 0$
$f_3 = 0$ $r_6 = 0$
$f_2 = 0$ $r_5 = 1$
$f_1 = 1$ $r_4 = 1$
$f_5 = 0$ $r_9 = 0$
$f_4 = 0$ $r_8 = 1$
$f_3 = 0$ $r_7 = 0$
$f_2 = 0$ $r_6 = 0$
$f_1 = 1$ $r_5 = 1$
$f_0 = 1$ $r_4 = 1$
$f_5 = 0$ $r_{10} = 0$
$f_4 = 0$ $r_9 = 0$
$f_3 = 0$ $r_8 = 1$
$f_2 = 0$ $r_7 = 0$
$f_1 = 1$ $r_6 = 0$
$f_0 = 1$ $r_5 = 1$
0 1 1 0 0 0
$\log n$



$f_5 = 0$ $r_9 = 0$
$f_5 = 0$ $r_{10} = 0$
$r_{11} = 0$
$f_4 = 0$ $r_6 = 0$
$f_4 = 0$ $r_7 = 0$
$f_4 = 0$ $r_8 = 1$
$f_4 = 0$ $r_9 = 0$
$f_3 = 0$ $r_5 = 1$
$f_3 = 0$ $r_6 = 0$
$f_3 = 0$ $r_7 = 0$
$f_3 = 0$ $r_8 = 1$
$f_2 = 0$ $r_4 = 1$
$f_2 = 0$ $r_5 = 1$
$f_2 = 0$ $r_6 = 0$
$f_2 = 0$ $r_7 = 0$
$f_1 = 1$ $r_3 = 1$
$f_1 = 1$ $r_4 = 1$
$f_1 = 1$ $r_5 = 1$
$f_1 = 1$ $r_6 = 0$
$f_0 = 1$ $r_4 = 1$
$f_0 = 1$ $r_5 = 1$
$f =$
$s =$
$\log n$


$f = 1\ 1\ 0\ 0\ 0\ 0$
$s = 1\ 0\ 0\ 0\ 1\ 1$
$f_4 = 0$ $r_6 = 0$
$f_3 = 0$ $r_5 = 1$
$f_2 = 0$ $r_4 = 1$
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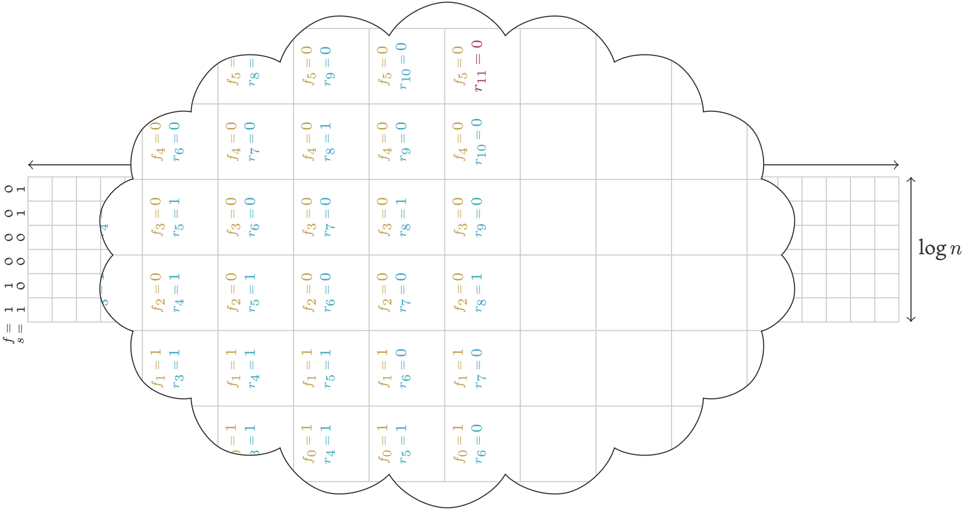
$f_3 = 0$ $r_8 = 1$
$f_2 = 0$ $r_7 = 0$
$f_1 = 1$ $r_6 = 0$
$f_0 = 1$ $r_5 = 1$
$r_{11} = 0$
$r_{10} = 0$
$r_9 = 0$
$r_8 = 1$
$r_7 = 0$
$r_6 = 0$
$\log n$



$f_4 = 0$ $r_6 = 0$
$f_3 = 0$ $r_5 = 1$
$f_2 = 0$ $r_4 = 1$
$f_1 = 1$ $r_3 = 1$
$f_4 = 0$ $r_7 = 0$
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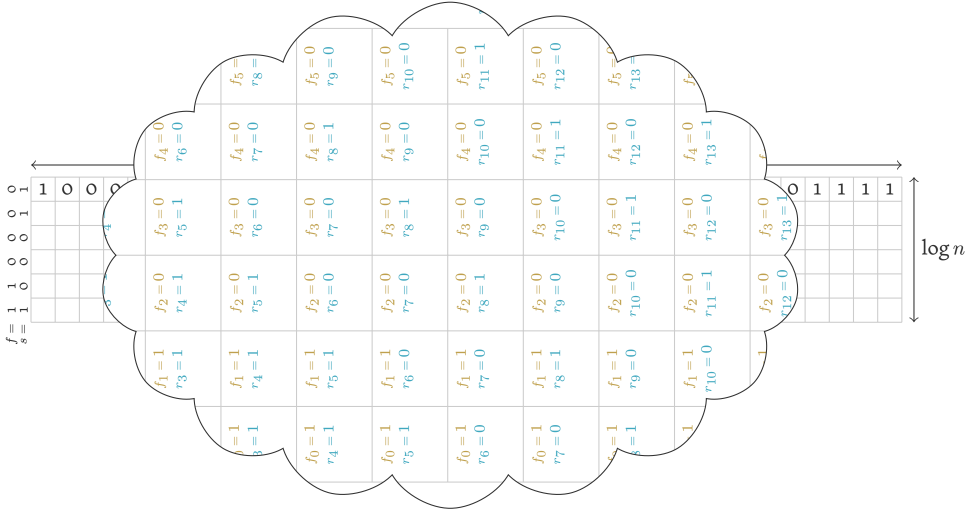
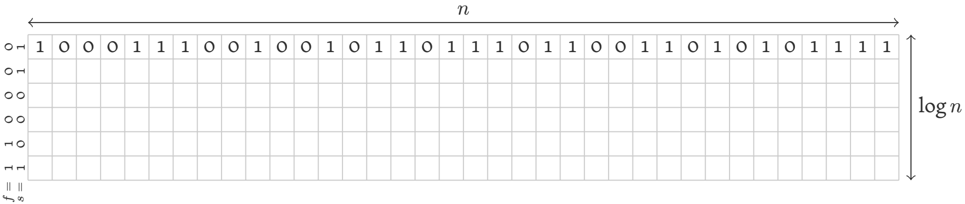
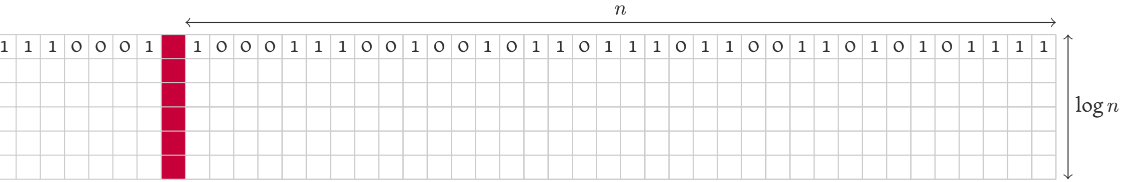
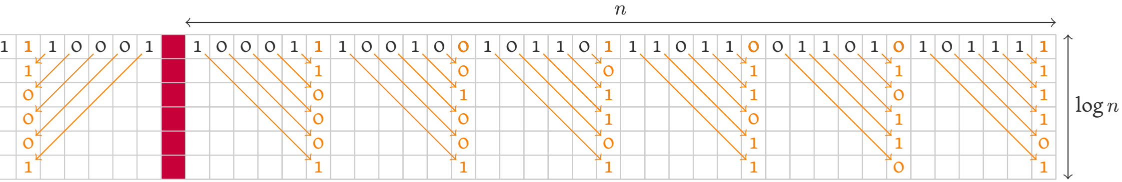
$f_2 = 0$ $r_8 = 1$
$f_1 = 1$ $r_7 = 0$
$f_0 = 1$ $r_6 = 0$
$\log n$

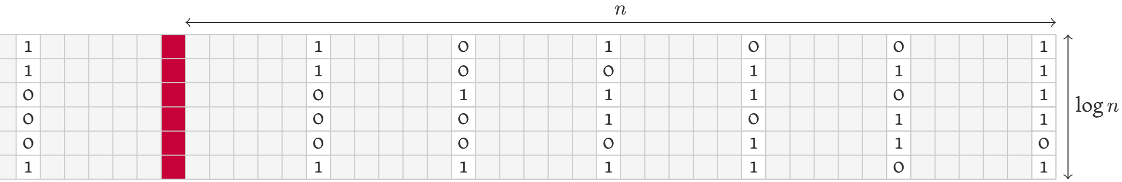

log n

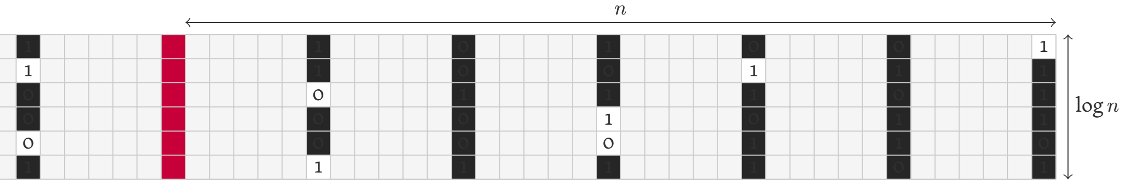

n
log n
1 0 0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 1 1 1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1
f = 1 1 0 0 0 0
s = 1 0 0 0 1 1

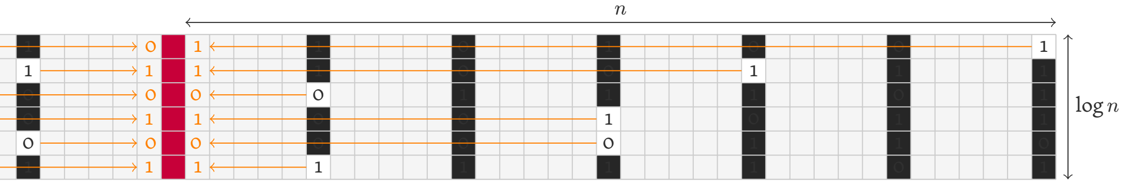

n
1 1 1 0 0 0 1
1 0 0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 1 1 1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1
log n



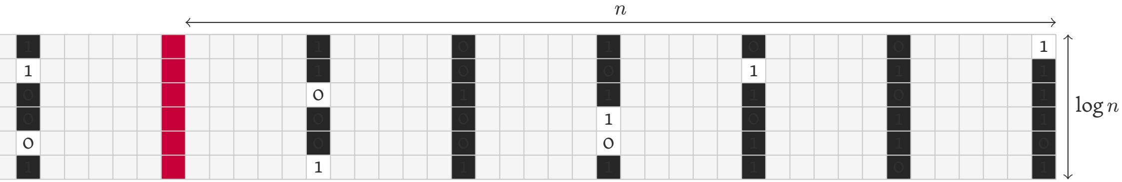
n
log n


n
$\log n$


n
$\log n$


n
$\log n$

Soficity of X_{NEQ}



Conclusion

If we put the columns at distance $\log n$ from each other, the tiling space X_{NEQ} becomes sofic.

Soficity of $X_{\mathbf{NEQ}}$: some remarks

By stacking blocks on top of each other, we solve $\textsc{NEQ}(n \times n)$ in a sofic manner:

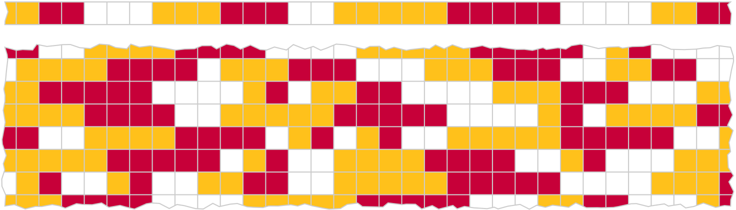
1. *Block encoding*: since $\log n^d = O(\log n)$, we also solve $\textsc{NEQ}(n^d \times n)$;
2. *Iterating the construction*: by piling this construction on top of itself, we can reduce the distance between columns to arbitrary $\log \log \ldots \log n$.
3. *Motivation*: Actually inspired by a question of Guillon and Jeandel:

Conjecture (Guillon & Jeandel, $\simeq$ 2015)

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a tiling space on $\mathbb{Z}$, and assume that $X^{\rightleftharpoons} \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is a $\mathbb{Z}^2$ sofic space

$$X^{\rightleftharpoons} = \{x \in \mathcal{A}^{\mathbb{Z}^2} : \forall n \in \mathbb{Z},\ x|_{\mathbb{Z}\times\{n\}} \in X\}.$$

Is X sofic on $\mathbb{Z}$?



Conclusion

Sofic tilings generalize regular languages into infinite and higher-dimensional words.

Main question

When is a given space $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ sofic?

We don't know! But: **communication complexity can help!**

- Sofic spaces can synchronize $O(n^{d-1})$ bits across the border of their $[\![1, n]\!]^d$ patterns;
- Inside a domain $[\![1, n]\!]^d$, a limited amount of computations can be performed.
  - About X_{NEQ}: are there "space-efficient" PRNG?
  - About X_{NEQ}: can we avoid moving the columns away from each other? (Maybe with a cellular automaton?)

Future work: open questions about multidimensional soficity.

- Jeandel's conjecture about the complexity of free extensions;
- Weiss' conjecture about entropies of local covers.


That's all Folks!