Information, communication and complexity of sofic tilings

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Joint work with Léo Paviet Salomon and Pascal Vanier

Journées postdocs du LIP, ENS de Lyon

24 March 2026

Shift spaces

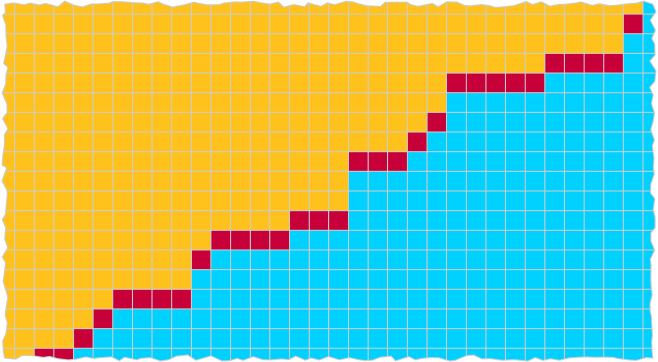
Definition

A *shift space* on $\mathbb{Z}^d$ is a set of colorings $\mathbb{Z}^d \to \mathcal{A}$ ("*tilings*") defined by forbidden patterns $\mathcal{F}$:

$$X_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall w \in \mathcal{F},\, w \text{ does not appear in } x \right\}.$$

Example:

$\mathcal{F} = \{$  $\}$



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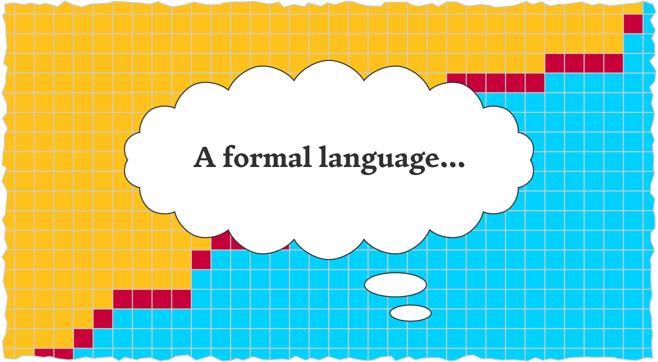

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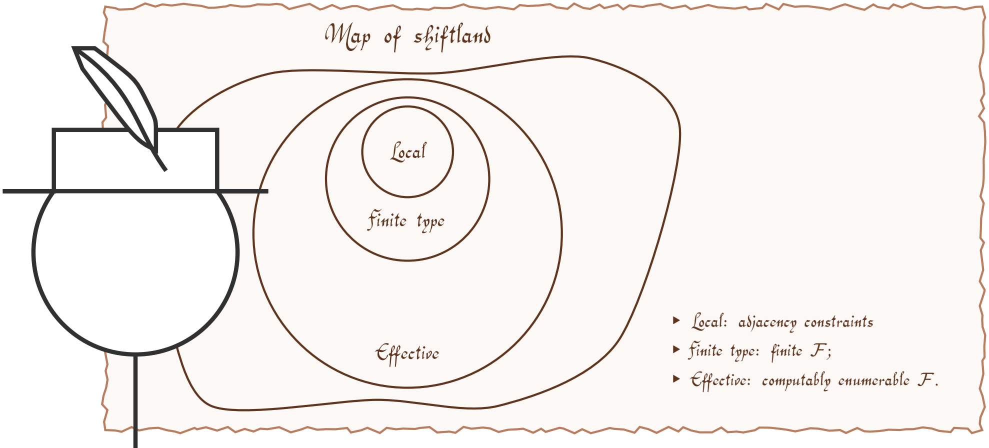

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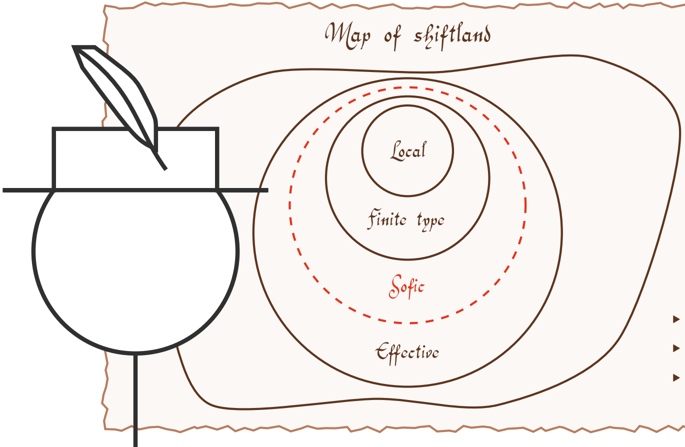

- Local: adjacency constraints
- Finite type: finite $\mathcal{F}$;
- Effective: computably enumerable $\mathcal{F}$.

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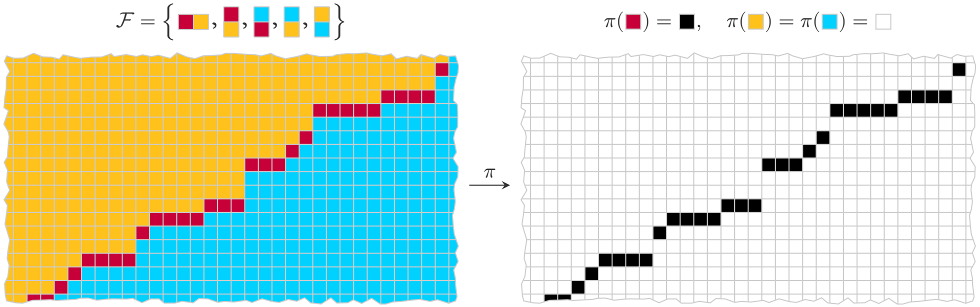
Definition (Sofic space, ≃ 1973)

A shift space is *sofic* if it can be defined as the cell-by-cell projection of a local space.



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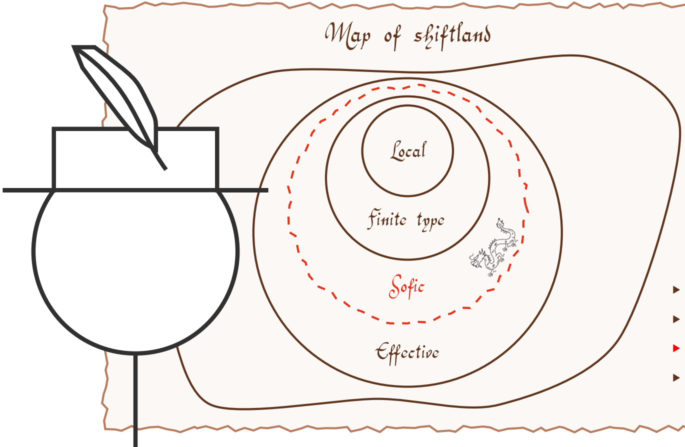

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Shift spaces on $\mathbb{Z}$

Shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$	**Language** $L \subseteq \mathcal{A}^*$
Local/finite type	Local
Sofic	Rational/regular
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Example

The sunny-side-up $X_{\blacksquare} \subseteq \{\square, \blacksquare\}^{\mathbb{Z}}$ is sofic.

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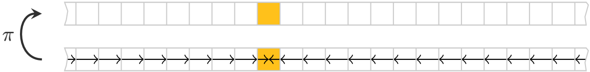

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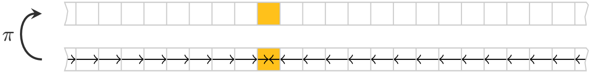

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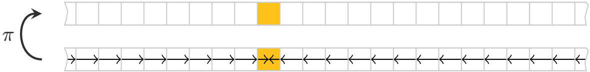

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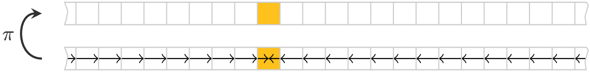

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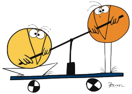

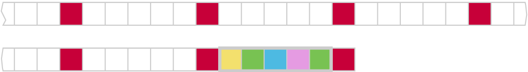

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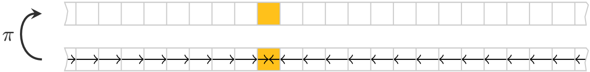

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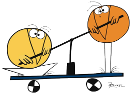

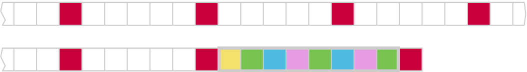

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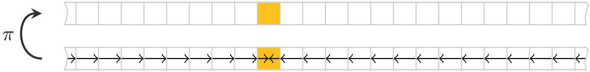

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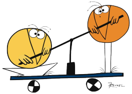

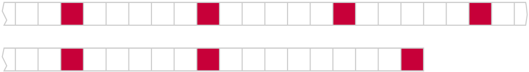

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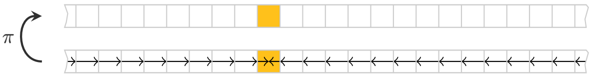

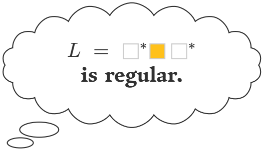

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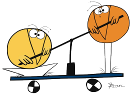

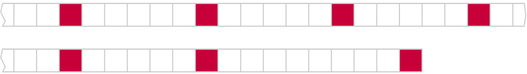

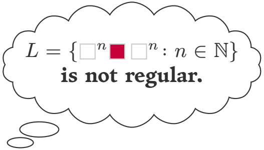

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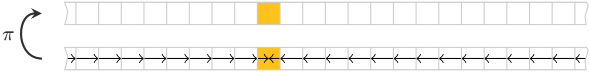

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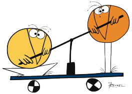

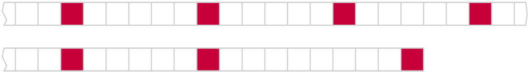

$L = \{\square^n \blacksquare \square^n : n \in \mathbb{N}\}$
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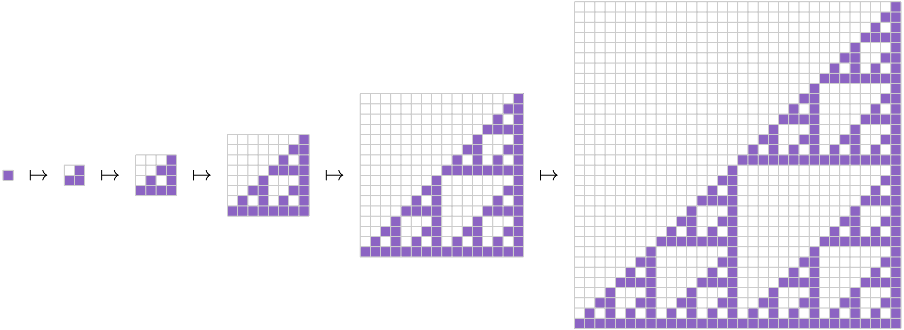
Lemma (Myhill-Nerode)

A shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is sofic if and only if it defines finitely many *extender sets* (*i.e.* "contexts").

Soficity of shift spaces on $\mathbb{Z}^d$: examples

- Substitutive shifts;
  [Mozes, 1989]



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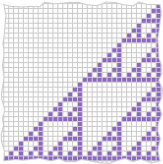

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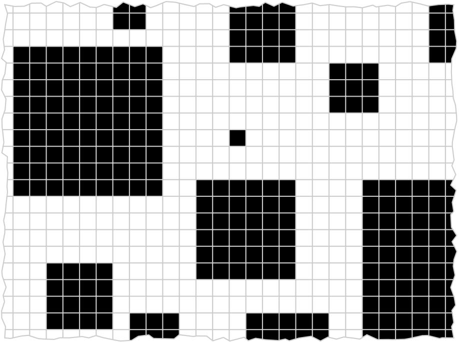

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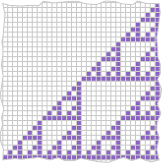

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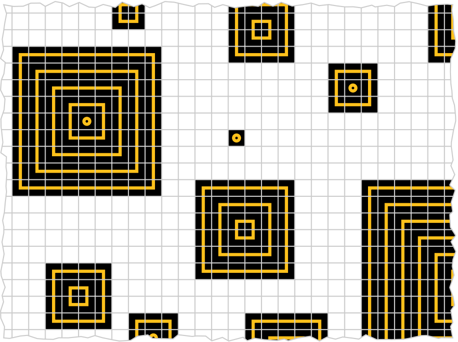

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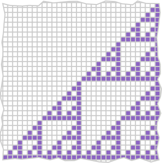
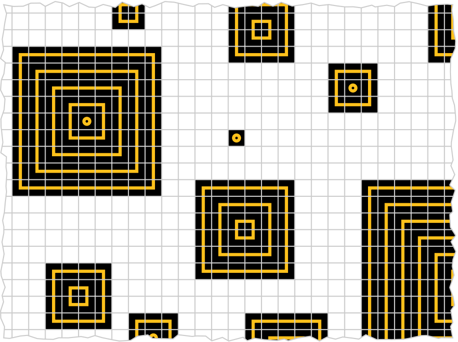


- Seas of squares (of Π_1^0 sizes);
  [Westrick, 2017]



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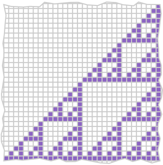

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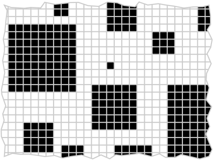

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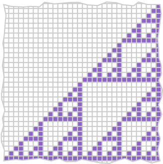

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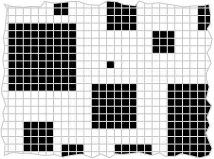

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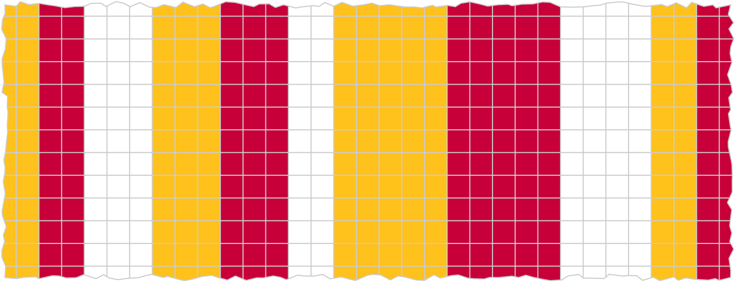

Soficity of shift spaces on $\mathbb{Z}^d$: expressiveness limits

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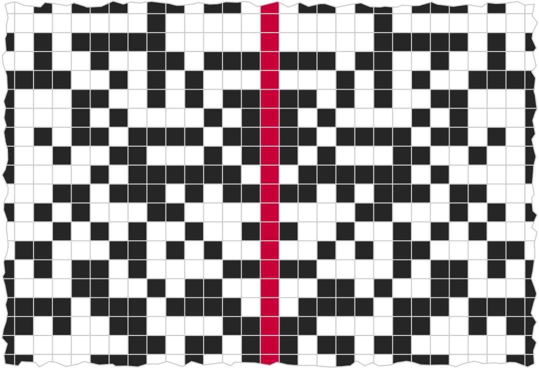
The mirror shift space is **not** sofic on $\mathbb{Z}^d$ for any $d \in \mathbb{N}$:



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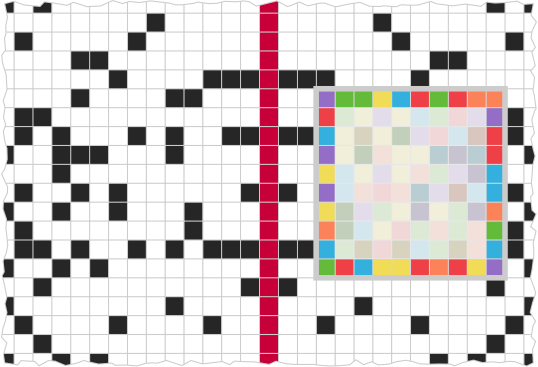

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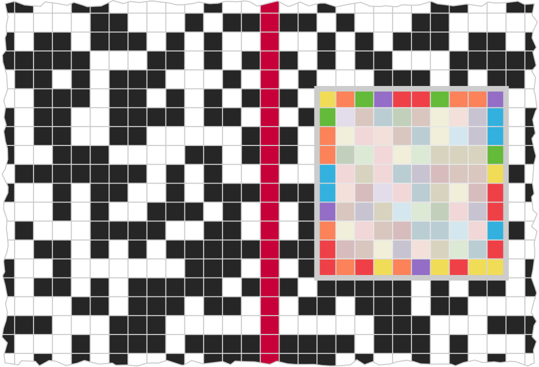

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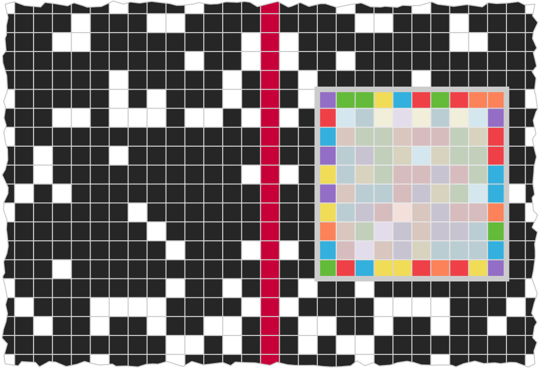

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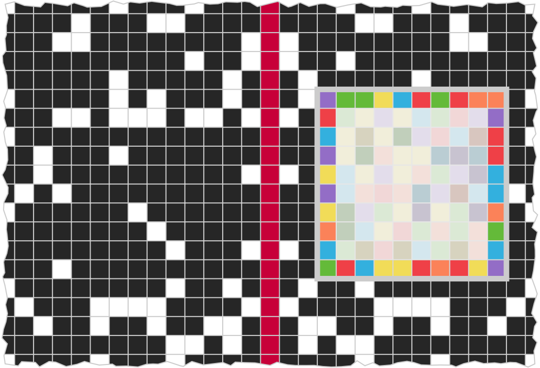

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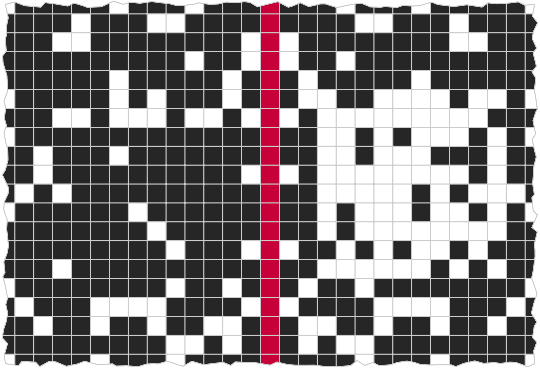

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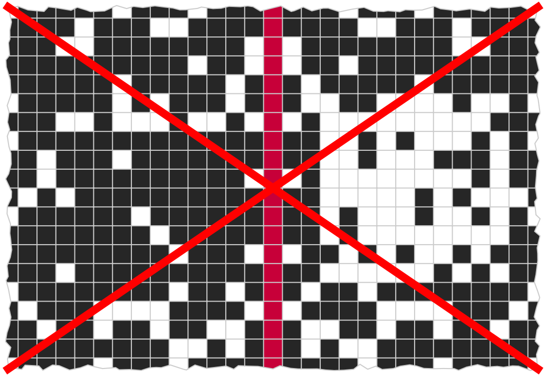

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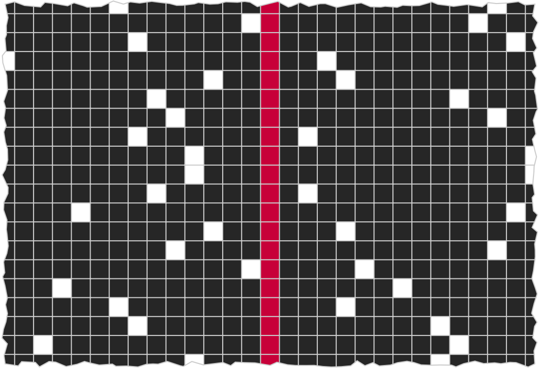
Intuition

Patterns of domain $[\![n]\!]^d$ in a sofic shift can only exchange $O(n^{d-1})$ bits with their exterior.

And most of the time...

Example

Is the shift X_{NEQ} sofic on $\mathbb{Z}^2$?

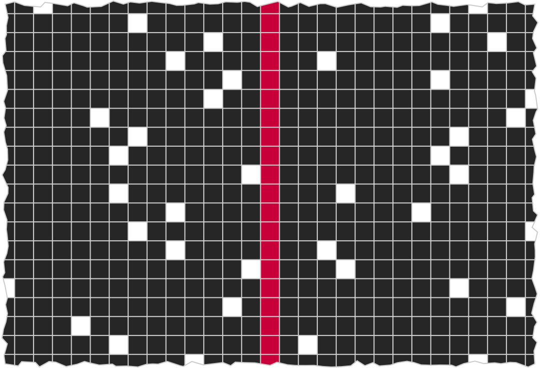


We don't know!

And most of the time...

Example

Is the shift X_{NEQ} sofic on $\mathbb{Z}^2$?



We don't know!

Communication complexity

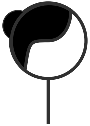
Communication complexity

Definition

Communication complexity quantifies the communication required to solve a distributed problem.

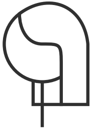
In a **communication problem**, we compute a problem $P\colon \{0,1\}^* \times \{0,1\}^* \to \{\top, \bot\}$.

Alice



$w_A \in \{0,1\}^*$

Bob



$w_B \in \{0,1\}^*$

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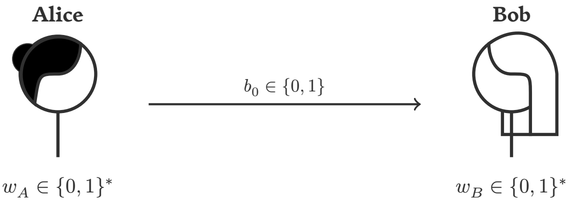

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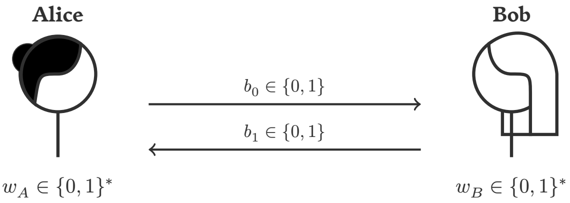

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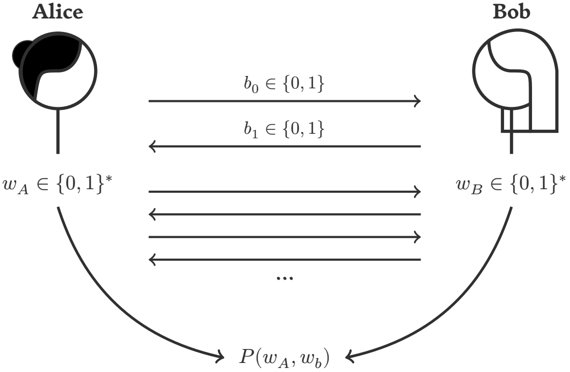


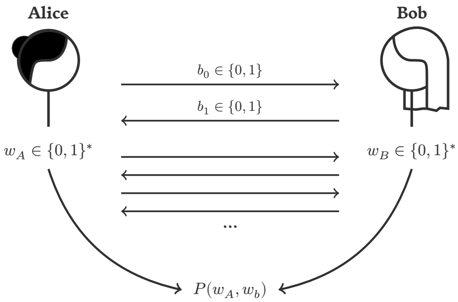
Communication complexity

Definition

For a problem P: $\{0,1\}^* \times \{0,1\}^* \to \{\top, \bot\}$, denote:

- $\mathcal{C}(P)$ the size[1] of a minimal protocol solving P;
- $\mathcal{NC}(P)$ the size of a minimal *non-deterministic*[2] protocol solving P.

[1] Where the *size* of a protocol is the length of its maximal transcript.
[2] If $P(w_A, w_B) = \top$, there must *exist* an accepting communication. Otherwise, all communications must reject.



Example

The problem EQ(n): for two words $w_A, w_B \in \{0,1\}^n$, do we have $w_A = w_B$?

$$\mathcal{C}(\mathrm{EQ}(n)) = \Theta(n);$$
$$\mathcal{NC}(\mathrm{EQ}(n)) = \Theta(n).$$

Example

The problem NEQ(n): for two words $w_A, w_B \in \{0,1\}^n$, do we have $w_A \neq w_B$?

$$\mathcal{C}(\mathrm{NEQ}(n)) = \Theta(n)(= \mathcal{C}(\mathrm{EQ}(n)));$$
$$\mathcal{NC}(\mathrm{NEQ}(n)) = \Theta(\log n).$$

Soficity and communication complexity

Lemma

In sofic spaces, pattern gluing of size n has *non-deterministic* communication complexity $O(n^{d-1})$.

Soficity and communication complexity

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Protocol of complexity $O(n^{d-1})$:

1. Guess a pattern cover;
2. Communicate the border;
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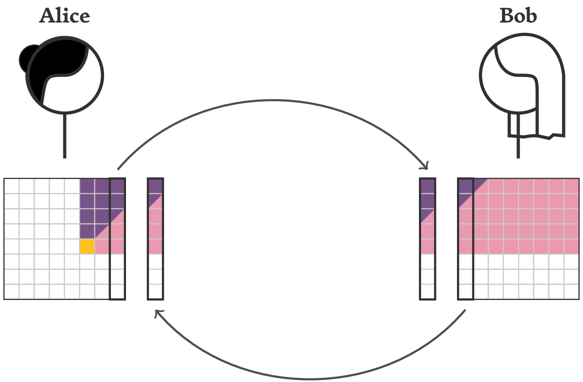

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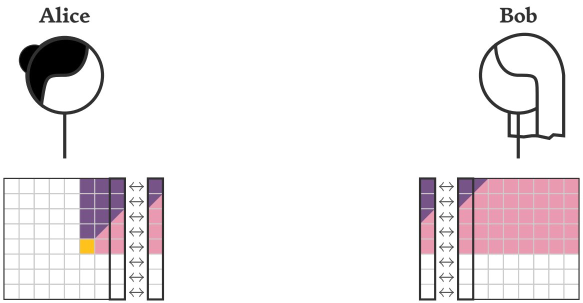

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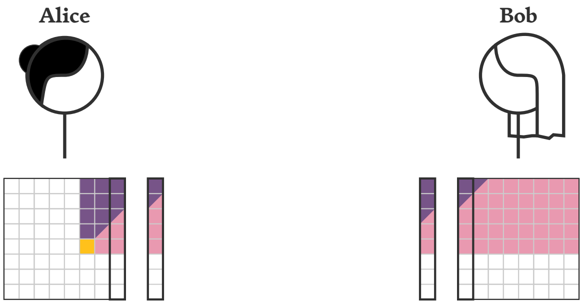

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On $\mathbb{Z}$, this is an equivalence!

Proposition ( )

A language $L \subseteq \mathcal{A}^*$ is iff $\mathcal{NC}(L) = O(1)$.[1]

[1] The problem L is the function $w_A, w_B \in \{0,1\}^* \times \{0,1\}^* \mapsto w_A \cdot w_B \overset{?}{\in} L$

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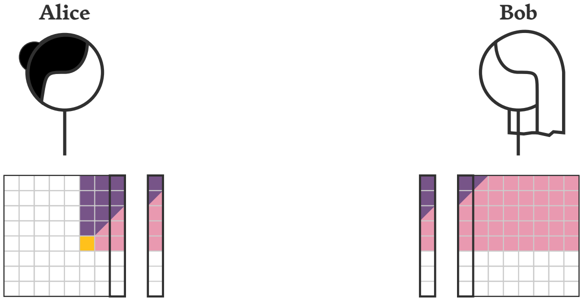

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Proposition ("Myhill-Nerode" revisited)

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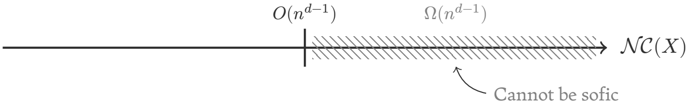
Soficity and communication complexity



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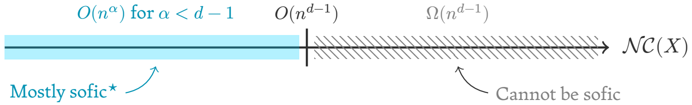
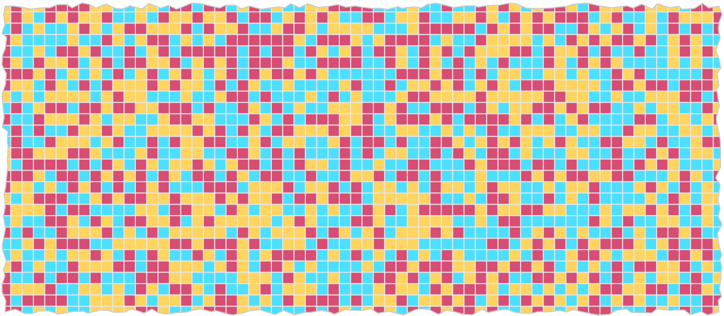


Theorem (C., Paviet Salomon, Vanier; 2026+)

Let $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ ($d \geq 2$) be an effective shift in which tiling validity can be checked inductively and:

- Communication steps are computable in time $O(n^{\alpha})$ for $\alpha < d-1$;

Then X is a sofic shift space. (Do not quote me on this!)



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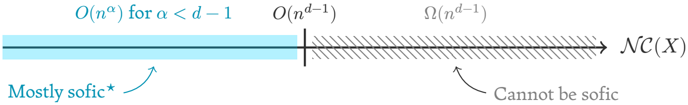

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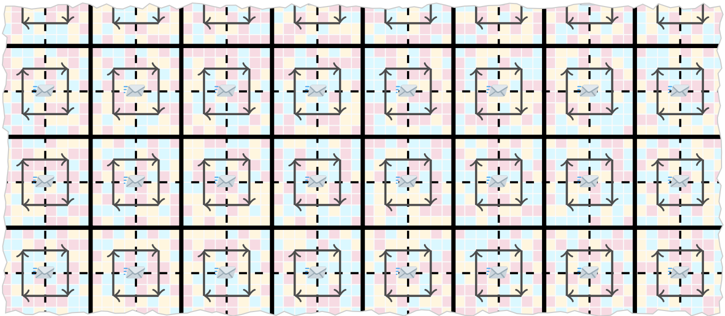

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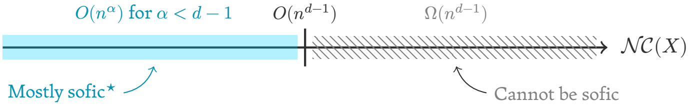

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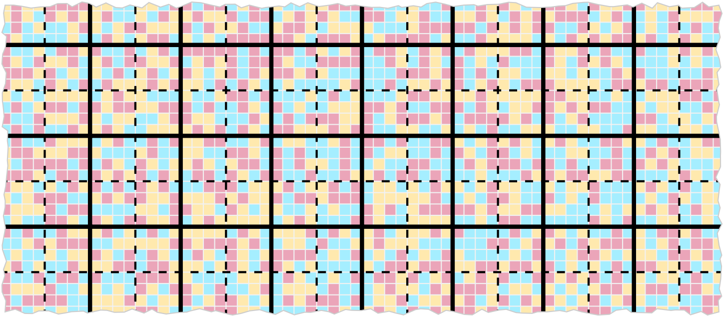

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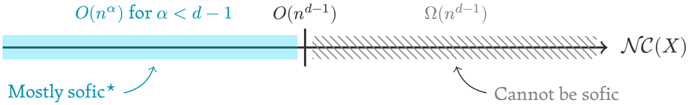

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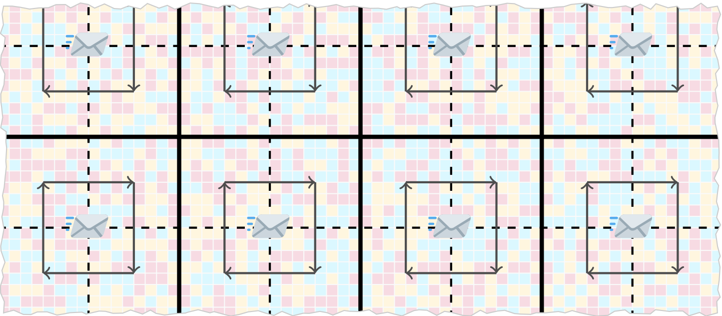

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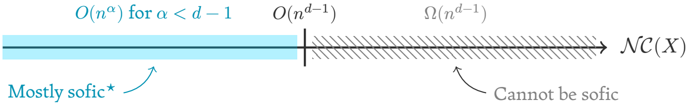

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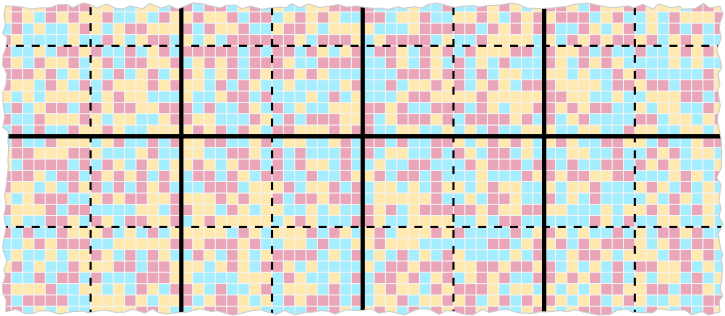

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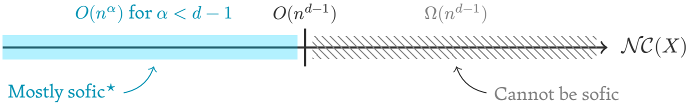

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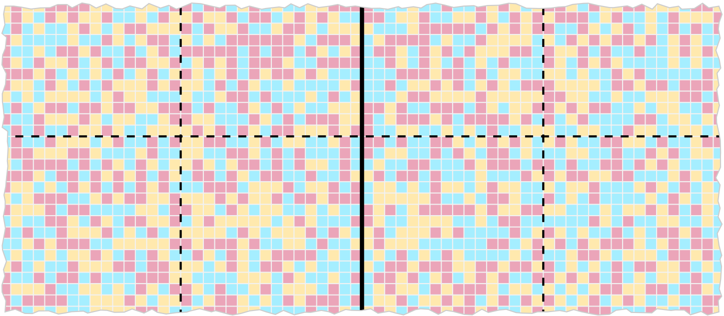

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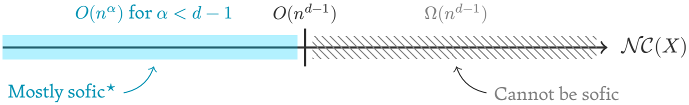

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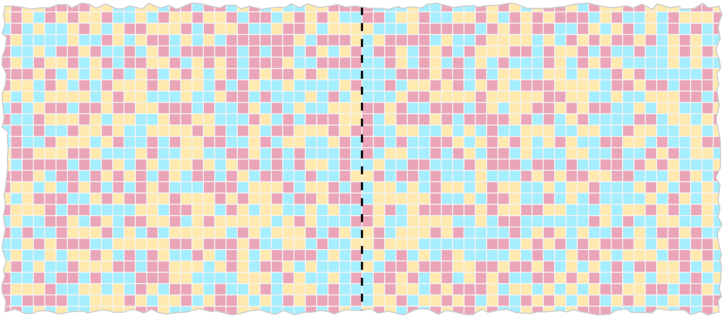

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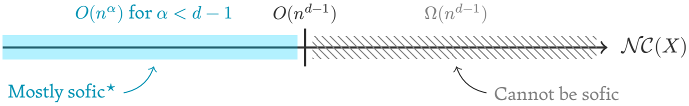

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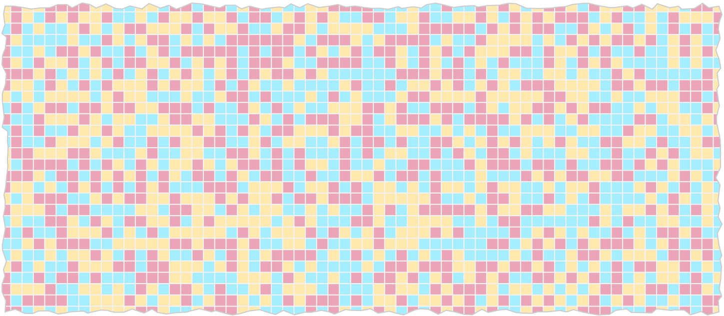

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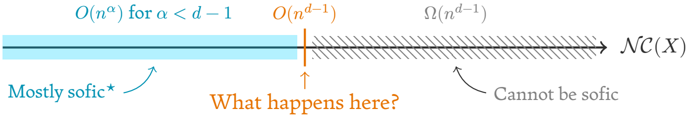

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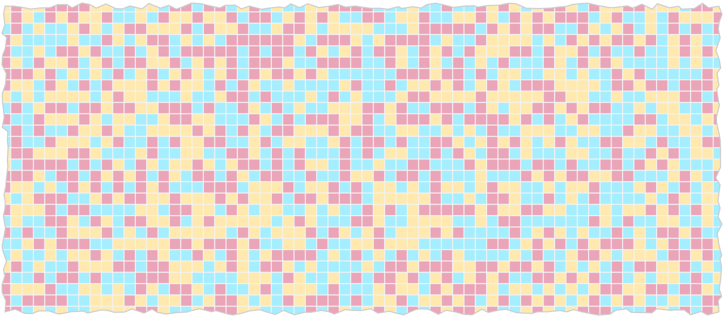
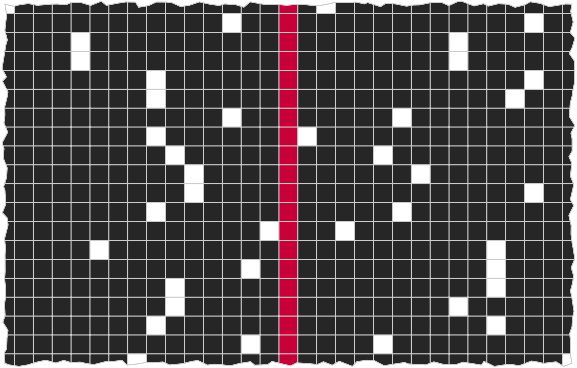


A toy example: the NEQ problem

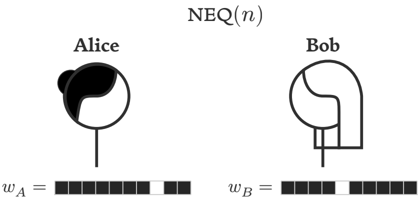
The NEQ problem



The NEQ problem

Definition (NEQ(n))

Define the communication problem NEQ$(n) = \{w_A, w_B \in \{0,1\}^n : |w_A|_1 = |w_B|_1 = 1 \text{ and } w_A \neq \overline{w_B}\}$.



Proposition

$\mathcal{NC}(\text{NEQ}(n)) = O(\log n)$...

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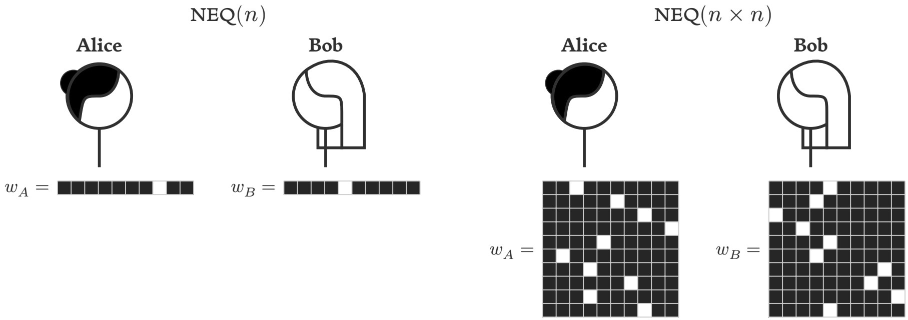

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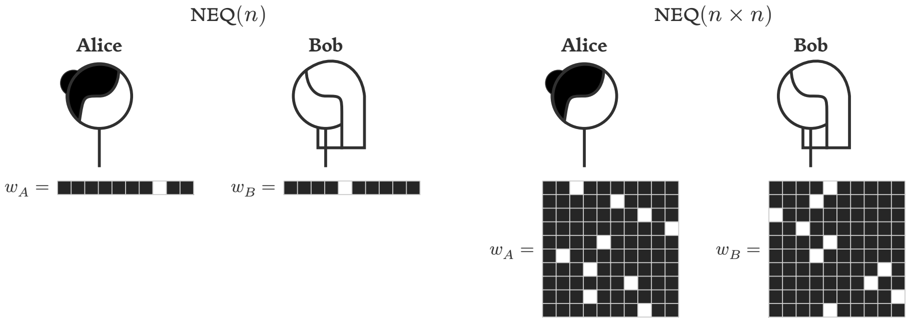

Proposition

$\mathcal{NC}(\text{NEQ}(n)) = O(\log n)$... but $\mathcal{NC}(\text{NEQ}(n \times n)) = O(n)$.

Question: Is the shift X_{NEQ} sofic on $\mathbb{Z}^2$?

A protocol for NEQ$(n \times n)$

Part 1: assuming shared randomness

$r = 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ \cdots$

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1. Draw a random $n \times n$ matrix M;
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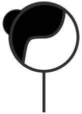
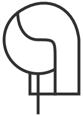
$r =$ 0 1 1 0 1 0 1 1 0 1 1 1 0 ⋯

Alice



Bob



$$M = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 0&1&0&0&0&1&1&0&1&1 \\ \hline 1&1&1&0&0&1&0&1&0&0 \\ \hline 1&0&0&0&1&1&1&1&0&1 \\ \hline 0&0&1&0&1&0&1&1&1&1 \\ \hline 1&0&1&0&1&1&1&0&1&0 \\ \hline 0&0&0&1&1&0&1&0&0&0 \\ \hline 1&0&0&1&1&0&0&0&0&1 \\ \hline 1&1&0&1&0&1&1&0&1&0 \\ \hline 0&1&0&1&1&1&0&1&1&0 \\ \hline 1&1&0&1&0&1&0&1&1&1 \\ \hline \end{array}$$

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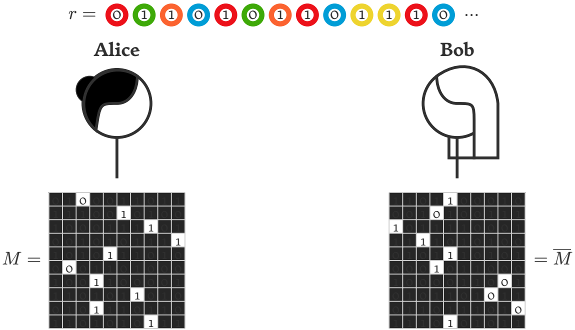

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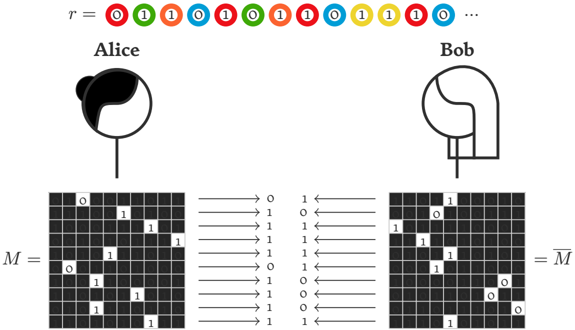

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Part 2: removing shared randomness

1. Communicate a non-deterministic seed $s \in \{0,1\}^*$;
2. GOTO [Part 1] using a pseudo-random generator.

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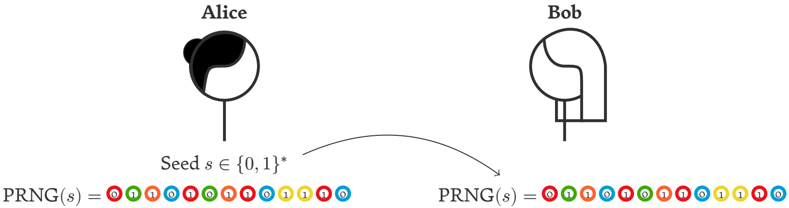
1. Draw a random $n \times n$ matrix M;
2. Alice computes the scalar products $a_i = \langle w_A|_{\{i\}\times\{1,\ldots,n\}}, M|_{\{i\}\times\{1,\ldots,n\}}\rangle \bmod 2$ (resp. Bob … $b_i = \ldots$);
3. Communicate $(a_i)_{1\leq i\leq n}$ and $(b_i)_{i\leq i\leq n}$, and check that all $a_i \neq b_i$.



Part 2: removing shared randomness

1. Communicate a non-deterministic seed $s \in \{0,1\}^*$;
2. `GOTO` [Part 1] using a pseudo-random generator.

A protocol for NEQ$(n \times n)$

Part 1: assuming shared randomness

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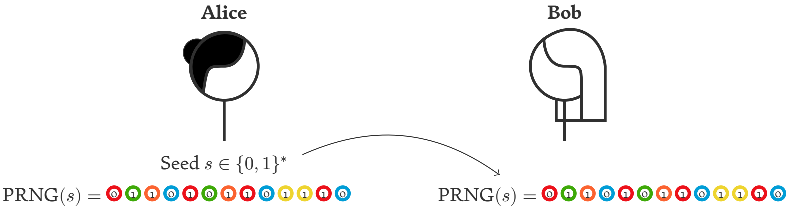

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A protocol for NEQ$(n \times n)$

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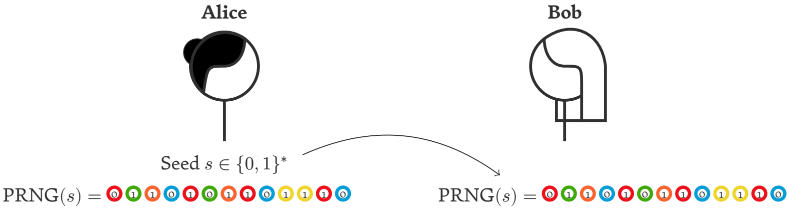
2. Alice computes the scalar products $a_i = \langle w_A|_{\{i\}\times\{1,\ldots,n\}}, M|_{\{i\}\times\{1,\ldots,n\}}\rangle \bmod 2$ (resp. Bob … $b_i = \ldots$);
3. Communicate $(a_i)_{1\le i\le n}$ and $(b_i)_{i\le i\le n}$, and check that all $a_i \neq b_i$. Communication $O(n)$ bits



Part 2: removing shared randomness

1. Communicate a non-deterministic seed $s \in \{0,1\}^*$; Communication $O(\log n)$ bits
2. `GOTO` [Part 1] using a pseudo-random generator.

Linear Feedback Shift Registers

Definition (LFSR)

Two words $(f, s) \in \{0,1\}^m \times \{0,1\}^m$ generate a *shift register sequence* $(r_k)_{k\in\mathbb{N}}$ defined by:

$$r_k = \begin{cases} s_k & \text{if k < m} \\ \langle f, r|_{k-m,\ldots,k-1} \rangle & \text{otherwise.} \end{cases}$$

Example

For $f = 0\,0\,0\,0\,1\,1\,0\,1\,1\,1$ and $s = 1\,1\,0\,1\,0\,0\,1\,1\,0\,0$:

$$r = 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0$$

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For $f = 0000110111$ and $s = 1101001100$:

$$r = 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1$$

$$0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1 \nearrow$$

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$$\qquad\qquad 0\;0\;0\;0\;1\;1\;0\;1\;1\;1 \nearrow$$

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$$\begin{array}{l} r = 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1 \\ \qquad\qquad\quad 0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\nearrow \end{array}$$

Linear Feedback Shift Registers

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Example

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$$r = 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0$$

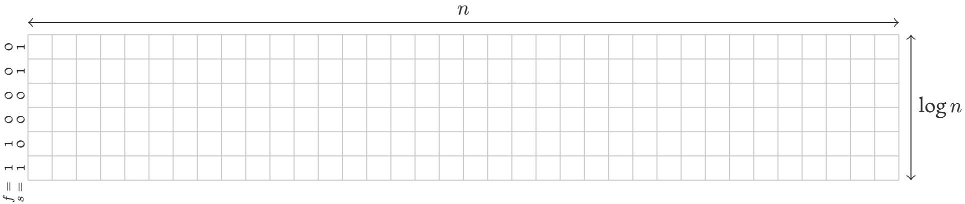
$$0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1$$

Lemma

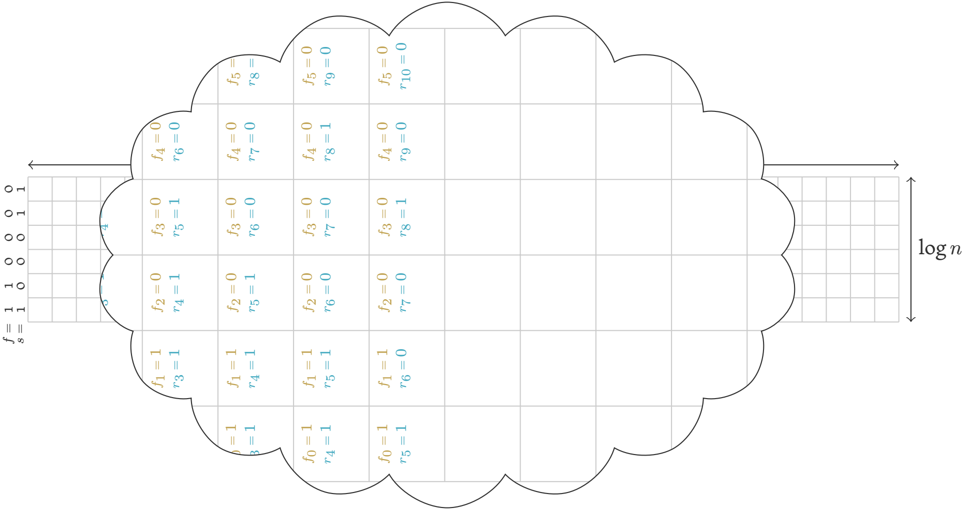
For every $m \in \mathbb{N}$ and indices $0 \le i_1 < \cdots < i_m < 2^m$, the set

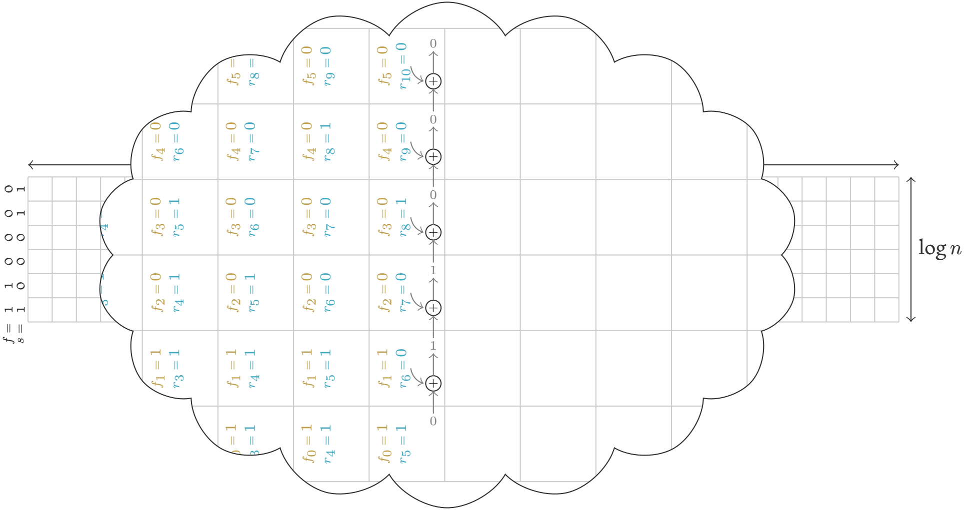
$$\{r|_{i_1,\dots,i_m} \in \{0,1\}^m : r \in \{0,1\}^{\mathbb{N}} \text{ generated by an LFSR of length } O(m)\}$$

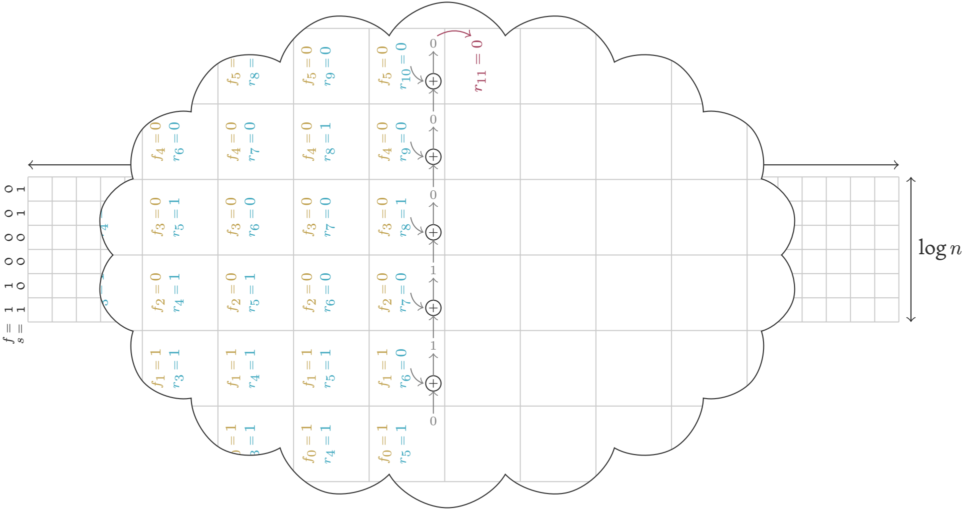
entirely covers $\{0,1\}^m$.

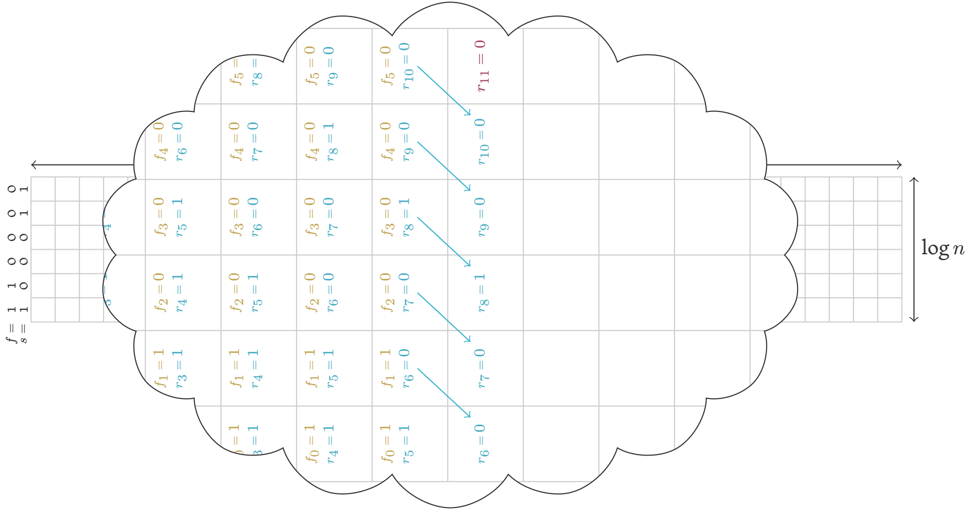


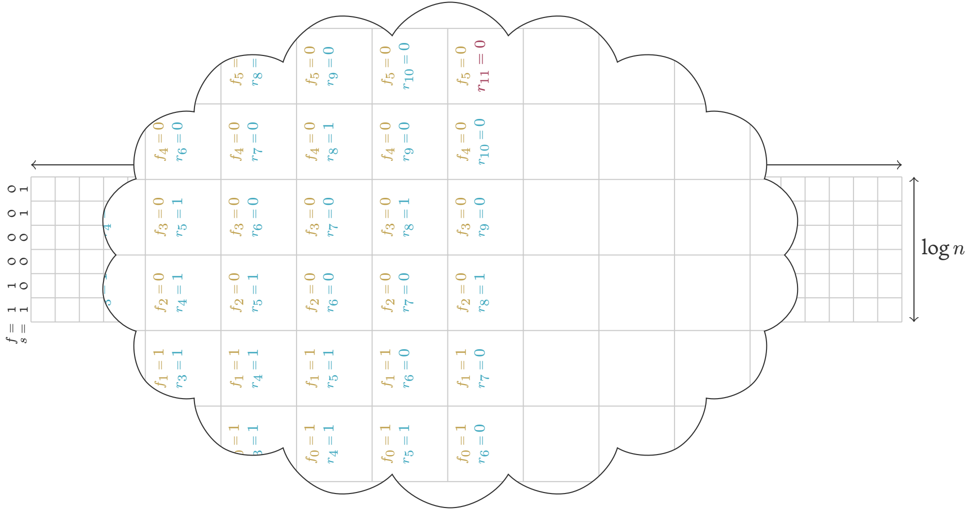
n
$\log n$
$f =$ 1 1 0 0 0 0
$s =$ 1 0 0 0 1 1

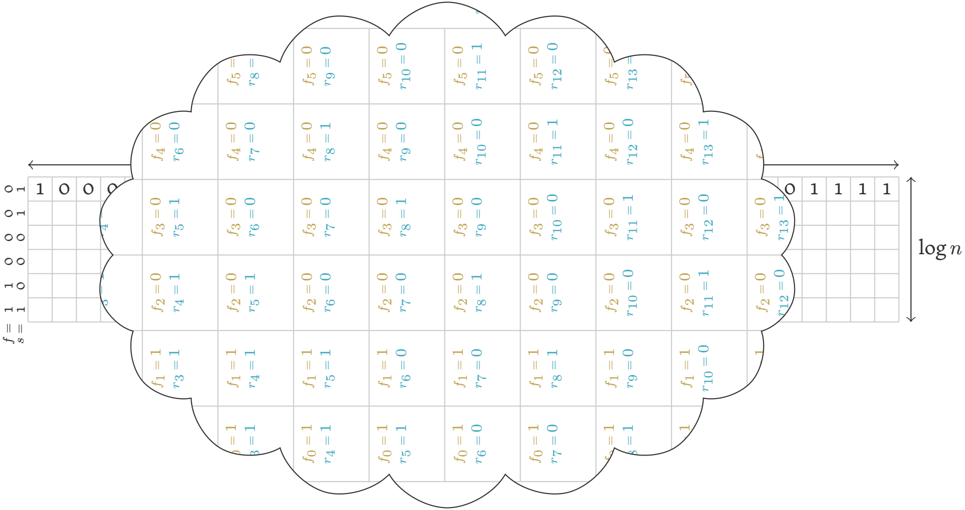

f =
s =
$f_4 = 0$
$r_6 = 0$
$f_3 = 0$
$r_5 = 1$
$f_2 = 0$
$r_4 = 1$
$f_1 = 1$
$r_3 = 1$
$f_4 = 0$
$r_7 = 0$
$f_3 = 0$
$r_6 = 0$
$f_2 = 0$
$r_5 = 1$
$f_1 = 1$
$r_4 = 1$
$f_5 = 0$
$r_9 = 0$
$f_4 = 0$
$r_8 = 1$
$f_3 = 0$
$r_7 = 0$
$f_2 = 0$
$r_6 = 0$
$f_1 = 1$
$r_5 = 1$
$f_0 = 1$
$r_4 = 1$
$f_5 = 0$
$r_{10} = 0$
$f_4 = 0$
$r_9 = 0$
$f_3 = 0$
$r_8 = 1$
$f_2 = 0$
$r_7 = 0$
$f_1 = 1$
$r_6 = 0$
$f_0 = 1$
$r_5 = 1$
$\log n$

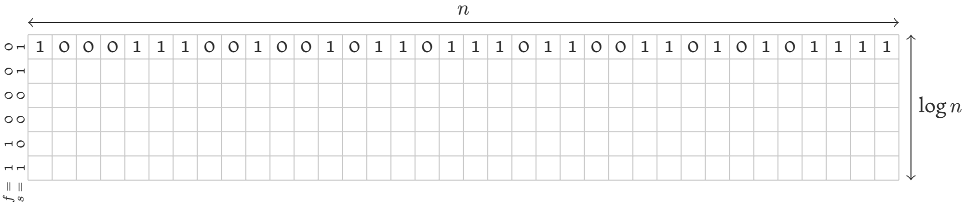

f = 1 1 0 0 0 0
s = 1 0 0 0 1 1
f_4 = 0 r_6 = 0
f_3 = 0 r_5 = 1
f_2 = 0 r_4 = 1
f_1 = 1 r_3 = 1
f_4 = 0 r_7 = 0
f_3 = 0 r_6 = 0
f_2 = 0 r_5 = 1
f_1 = 1 r_4 = 1
f_5 = 0 r_9 = 0
f_4 = 0 r_8 = 1
f_3 = 0 r_7 = 0
f_2 = 0 r_6 = 0
f_1 = 1 r_5 = 1
f_0 = 1 r_4 = 1
f_5 = 0 r_10 = 0
f_4 = 0 r_9 = 0
f_3 = 0 r_8 = 1
f_2 = 0 r_7 = 0
f_1 = 1 r_6 = 0
f_0 = 1 r_5 = 1
0
0
0
0
1
1
0
log n

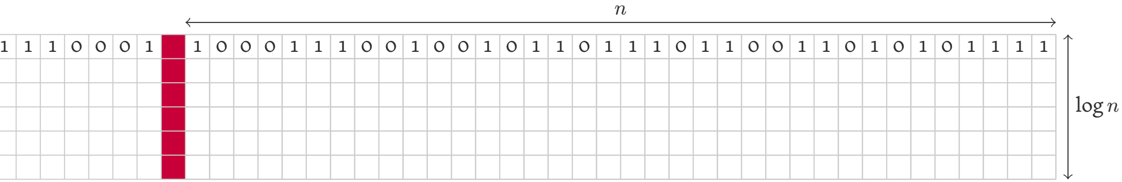

$f =$ 1 1 0 0 0 0
$s =$ 1 0 0 0 1 1
$f_4 = 0$ $r_6 = 0$
$f_3 = 0$ $r_5 = 1$
$f_2 = 0$ $r_4 = 1$
$f_1 = 1$ $r_3 = 1$
$f_4 = 0$ $r_7 = 0$
$f_3 = 0$ $r_6 = 0$
$f_2 = 0$ $r_5 = 1$
$f_1 = 1$ $r_4 = 1$
$f_5 = 0$ $r_9 = 0$
$f_4 = 0$ $r_8 = 1$
$f_3 = 0$ $r_7 = 0$
$f_2 = 0$ $r_6 = 0$
$f_1 = 1$ $r_5 = 1$
$f_0 = 1$ $r_4 = 1$
$f_5 = 0$ $r_{10} = 0$
$f_4 = 0$ $r_9 = 0$
$f_3 = 0$ $r_8 = 1$
$f_2 = 0$ $r_7 = 0$
$f_1 = 1$ $r_6 = 0$
$f_0 = 1$ $r_5 = 1$
$r_{11} = 0$
$\log n$

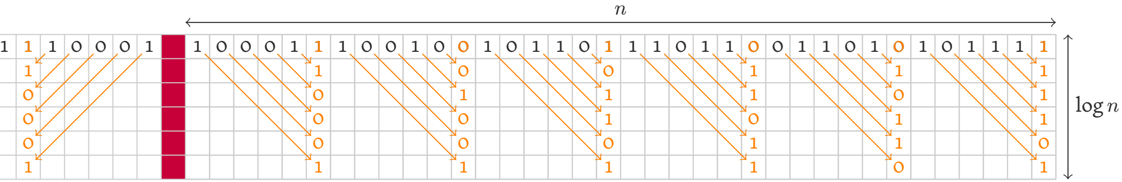

log n


f = 1 1 0 0 0 0
s = 1 0 0 0 1 1
log n

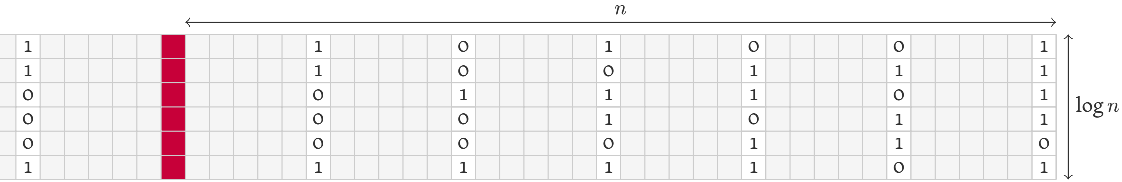

1 0 0 0
0 1 1 1 1
log n
f =
s =

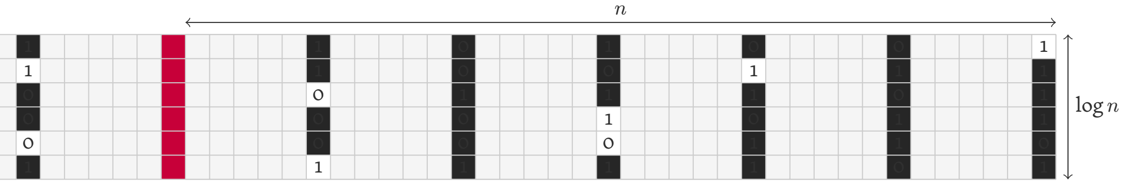

n
1 0 0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 1 1 1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1
log n
f = 1 1 0 0 0 0
s = 1 0 0 0 1 1

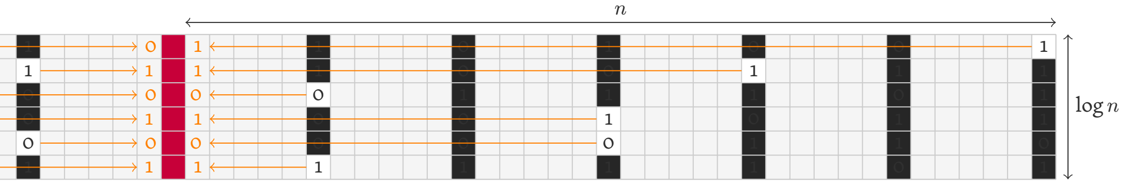

n
1 1 1 0 0 0 1
1 0 0 0 1 1 1 0 0 1 0 0 1 0 1 1 0 1 1 1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1
$\log n$



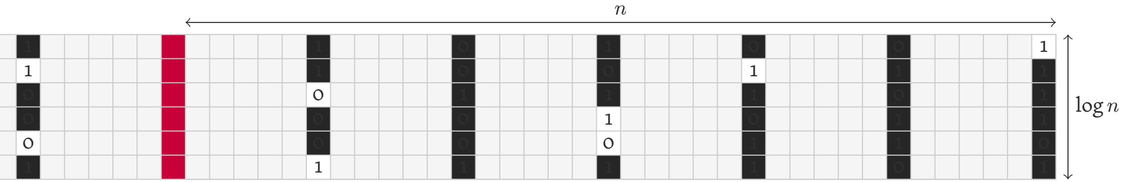
n
$\log n$


n
1 1 0 1 0 0 1
1 1 0 0 1 1 1
0 0 1 1 1 0 1
0 0 0 1 0 1 1
0 0 0 0 1 1 0
1 1 1 1 1 0 1
log n


n
log n


n
log n

Soficity of X_{NEQ}



Conclusion

If we put the columns at distance $\log n$ from each other, the shift X_{NEQ} becomes sofic.

Soficity of $X_{\mathbf{NEQ}}$: some remarks

By stacking blocks on top of each other, we solve $\textsc{NEQ}(n \times n)$ in a sofic manner:

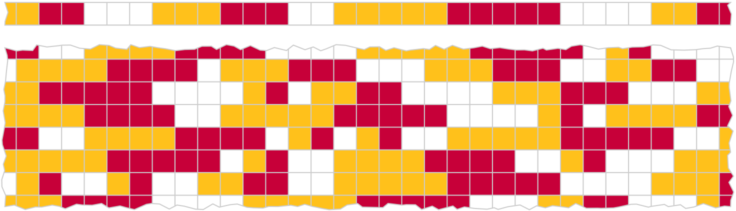
1. *Block encoding*: since $\log n^d = O(\log n)$, we also solve $\textsc{NEQ}(n^d \times n)$;
2. *Iterating the construction*: by piling this construction on top of itself, we can reduce the distance between columns to arbitrary $\log \log \ldots \log n$.
3. *Motivation*: Actually inspired by a question of Guillon and Jeandel:

Conjecture (Guillon & Jeandel, ≃ 2015)

Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a shift space on $\mathbb{Z}$, and assume that $X^{\rightleftharpoons} \subseteq \mathcal{A}^{\mathbb{Z}^2}$ is a $\mathbb{Z}^2$ sofic space

$$X^{\rightleftharpoons} = \{x \in \mathcal{A}^{\mathbb{Z}^2} : \forall n \in \mathbb{Z},\ x|_{\mathbb{Z}\times\{n\}} \in X\}.$$

Is X sofic on $\mathbb{Z}$?



Conclusion

**Sofic tilings generalize regular languages
into infinite and higher-dimensional words.**

Main question

When is a given space $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ sofic?

We don't know! But: **communication complexity can help!**

- Sofic spaces can synchronize $O(n^{d-1})$ bits across the border of their $[\![1,n]\!]^d$ patterns;
- Inside a domain $[\![1,n]\!]^d$, a limited amount of computations can be performed.
  - About X_{NEQ}: are there "space-efficient" PRNG?
  - About X_{NEQ}: can we avoid moving the columns away from each other? (Maybe with a cellular automaton?)

Future work: open questions about multidimensional soficity.

- Jeandel's conjecture about the complexity of free extensions;
- Weiss' conjecture about entropies of local covers.


That's all Folks!