Self-simulation, substitutions and soficity of multidimensional shift spaces

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Joint work with Léo Paviet Salomon and Pascal Vanier

Séminaire ESCAPE, LIRMM

03 February 2026

Shift spaces

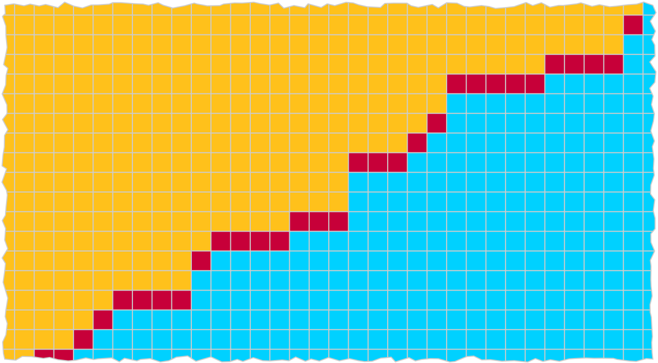
Definition

A *shift space* on $\mathbb{Z}^d$ is a set of colorings $\mathbb{Z}^d \to \mathcal{A}$ ("*configurations*") defined by forbidden patterns $\mathcal{F}$:

$$X_{\mathcal{F}} = \left\{ x \in \mathcal{A}^{\mathbb{Z}^d} : \forall w \in \mathcal{F},\ w \text{ does not appear in } x \right\}.$$

Example:

$\mathcal{F} = \Big\{$  $\Big\}$



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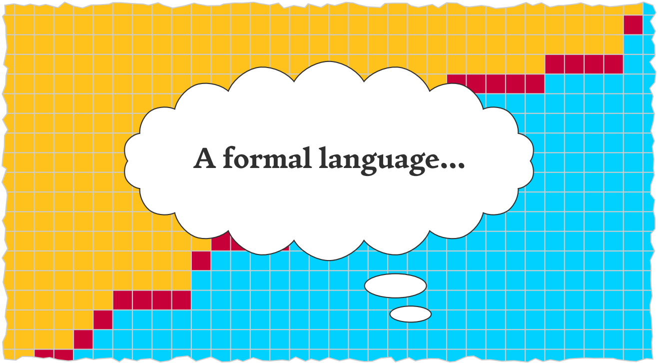

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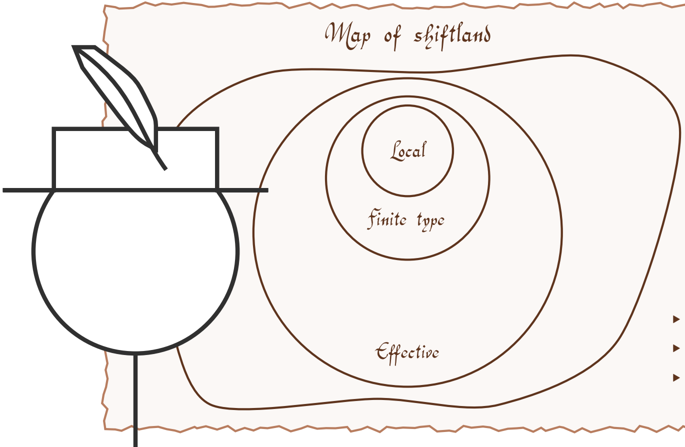

- Local: adjacency constraints
- Finite type: finite $\mathcal{F}$;
- Effective: computably enumerable $\mathcal{F}$.

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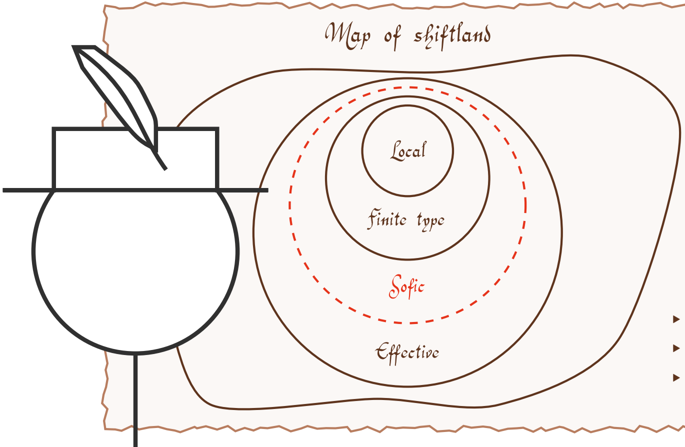

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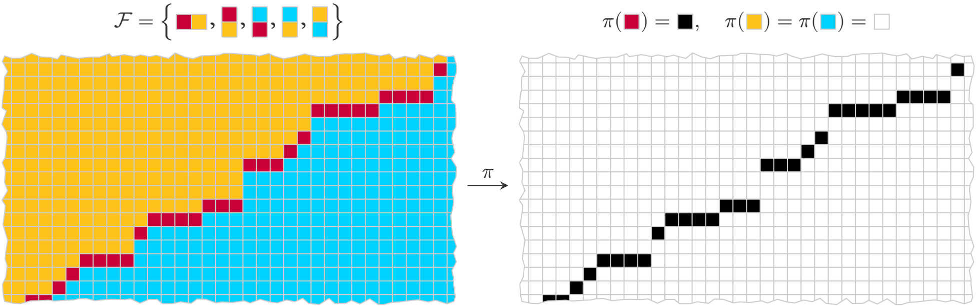
Definition (Sofic space, ≃ 1973)

A shift space is *sofic* if it can be defined as the cell-by-cell projection of a local space.



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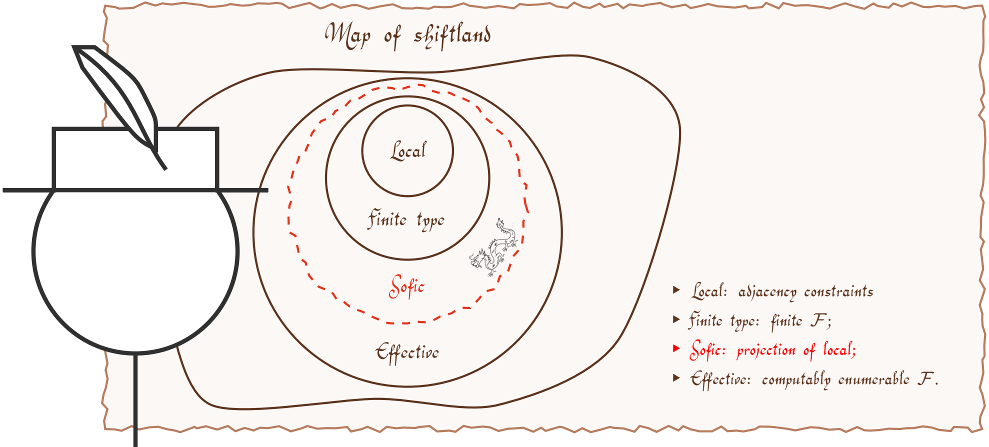

Shift spaces on $\mathbb{Z}$

Shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$	**Language** $L \subseteq \mathcal{A}^*$
Local/finite type	Local
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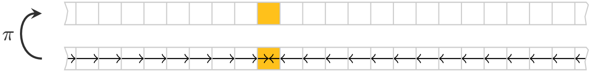

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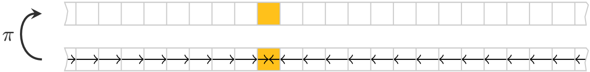

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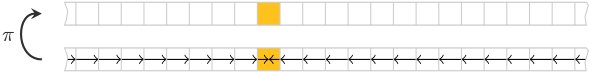

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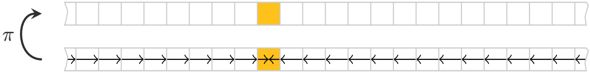

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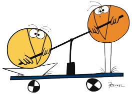

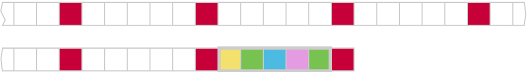

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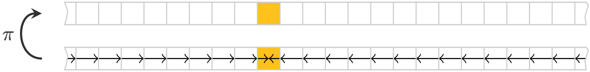

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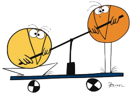

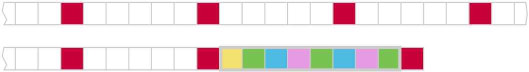

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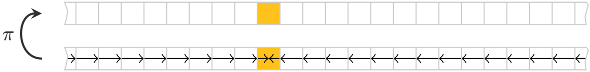

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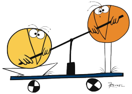

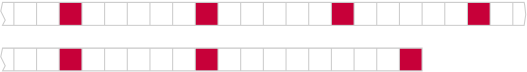

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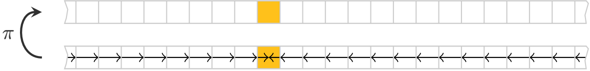

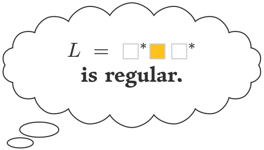

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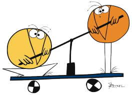

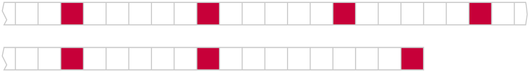

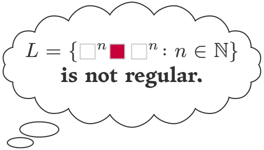

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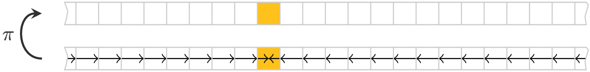

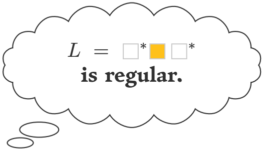

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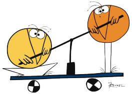
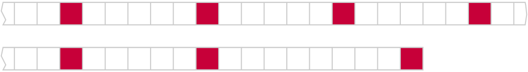
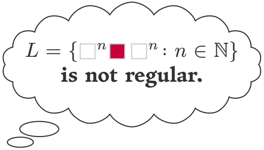






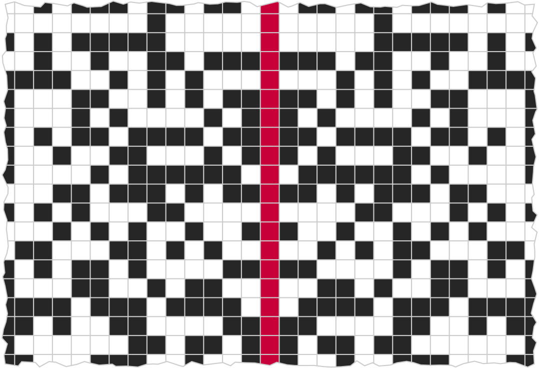
Lemma (Myhill-Nerode)

A shift space $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is sofic if and only if it defines finitely many *extender sets* (*i.e.* “contexts”).

Soficity of shift spaces on $\mathbb{Z}^d$: information bounds

Example

The mirror shift space is **not** sofic on $\mathbb{Z}^d$ for any $d \in \mathbb{N}$:



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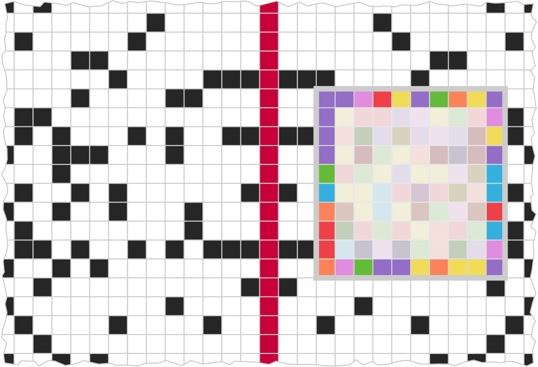

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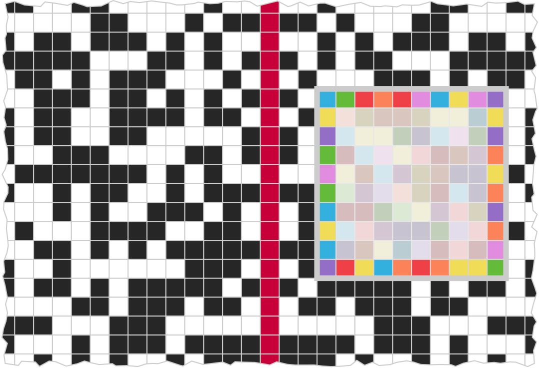

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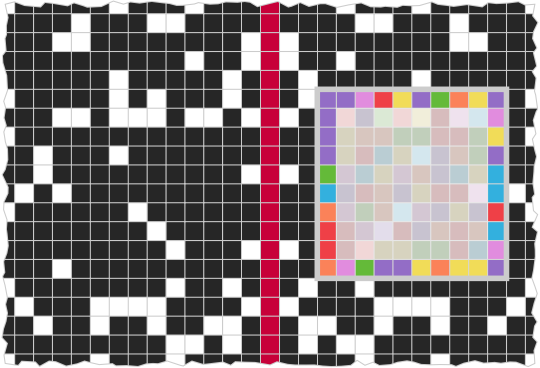

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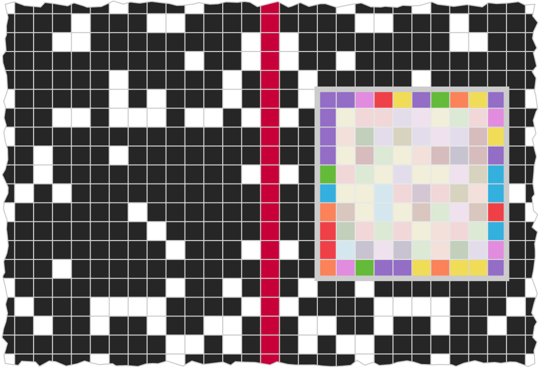

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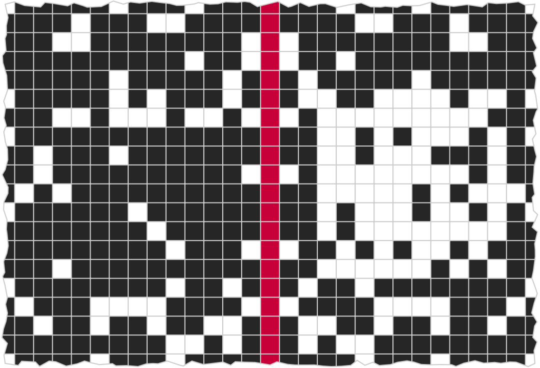

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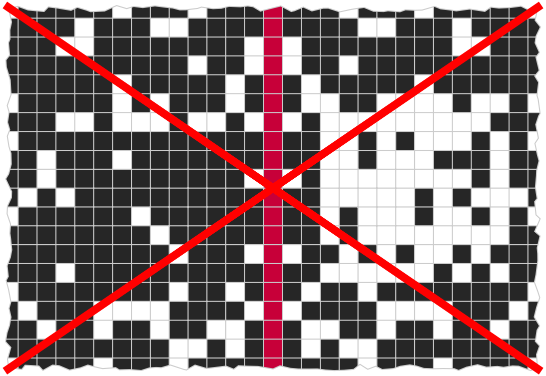

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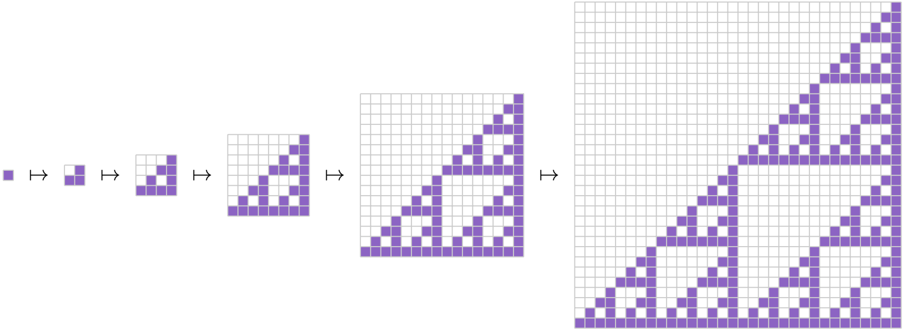
Intuition

Patterns of domain $[\![n]\!]^d$ in a sofic shift can only exchange $O(n^{d-1})$ bits with their exterior.

Soficity of shift spaces on $\mathbb{Z}^d$: examples

- Substitutive shifts;
  [Mozes, 1989]



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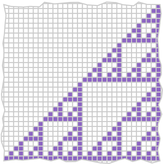
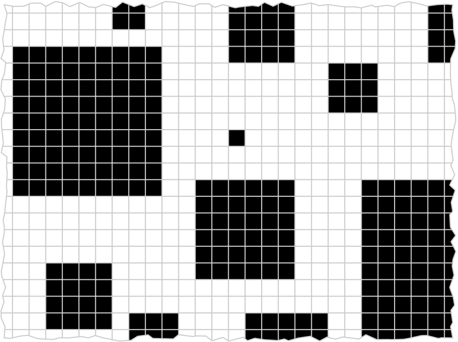


- Seas of squares;



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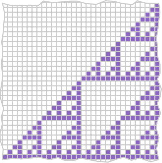

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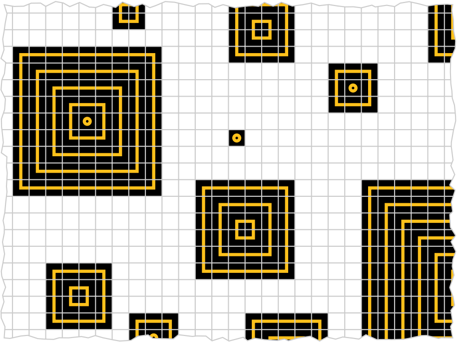

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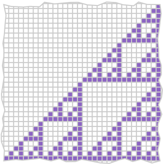
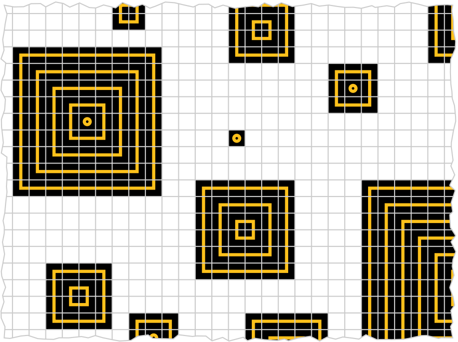


- Seas of squares (of Π_1^0 sizes);
  [Westrick, 2017]



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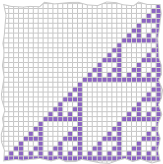

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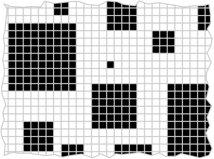

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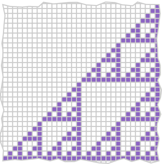

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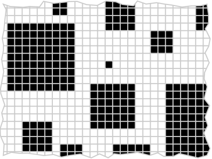

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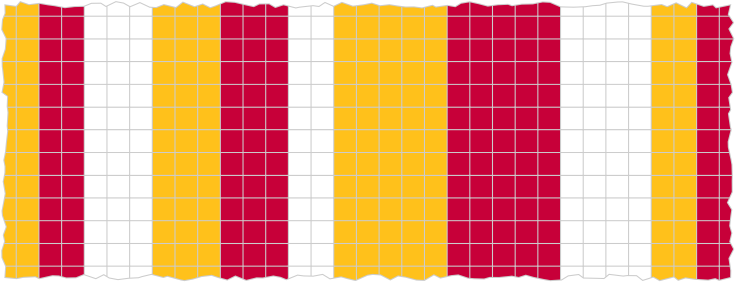
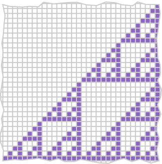
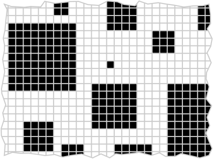


Soficity of shift spaces on $\mathbb{Z}^d$: examples

- Substitutive shifts;
  [Mozes, 1989]



- Seas of squares (of Π_1^0 sizes);
  [Westrick, 2017]



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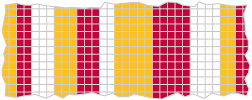


Fact

All these shifts can be proved sofic on $\mathbb{Z}^d$ $(d \geq 2)$ using the *fixed point construction*.

[Durand, Romashchenko & Shen, 2008+]

Sofic realization of s-adic limit spaces by self-simulation

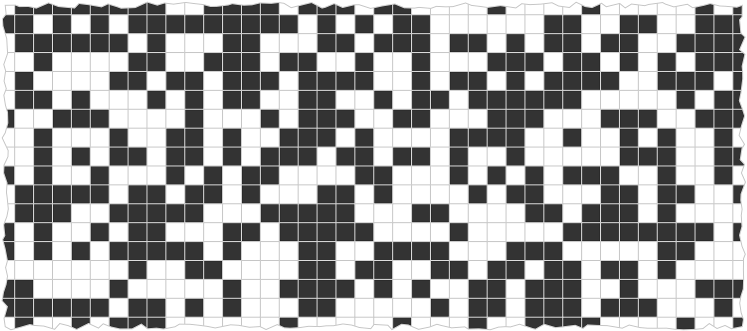
Substitutions

Definition

Given two alphabets $\mathcal{A}$, $\mathcal{B}$, and a zoom $N \in \mathbb{N}$, a *simulation* is an injective map

$$\mathcal{S}\colon \mathcal{A}^{[\![N]\!]^d} \rightrightarrows \mathcal{B}.$$

A simulation $\mathcal{S}$ induces a relation between configurations:



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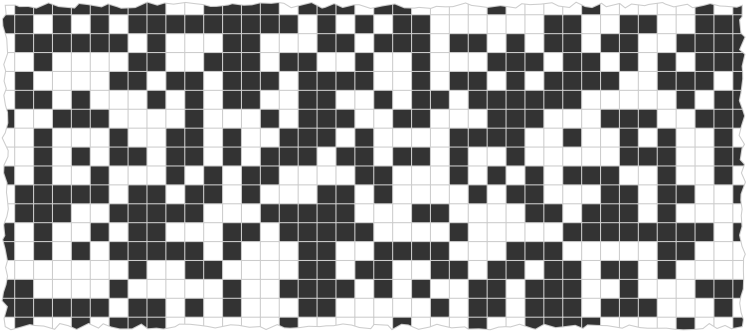

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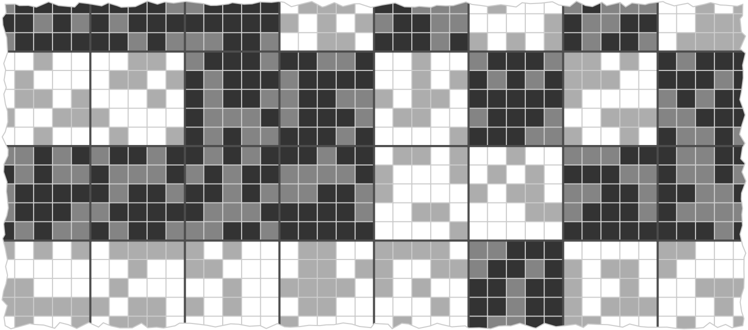

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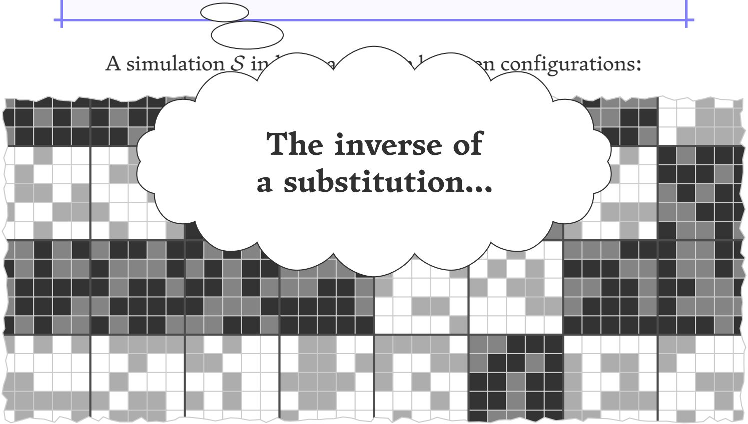

Main statement

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For a set of colors $C \subseteq \{0,1\}^*$, a *Wang tile* is some $t \in C^{2d+1}$:

A set of tiles $T \subseteq C^{2d+1}$ defines a local shift space X_T.

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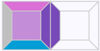

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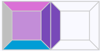

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is a sofic shift.

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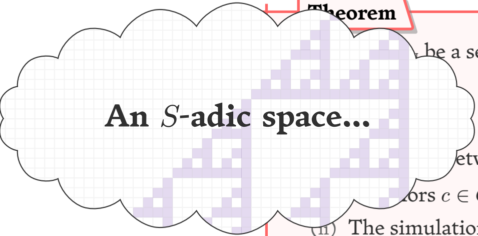
A sequence of substitutions $(\tau_\ell)_{\ell\in\mathbb{N}}$ (with $\tau_\ell\colon \mathcal{A}_{\ell+1} \rightrightarrows \mathcal{A}_\ell^{[\![N_\ell]\!]^d}$) defines an *$S$-adic limit space*

$$\left\{x \in \mathcal{A}_0^{\mathbb{Z}^d} : x = x_0 \xleftarrow{\tau_0} x_1 \xleftarrow{\tau_1} x_2 \xleftarrow{\tau_2} \dots\right\}$$



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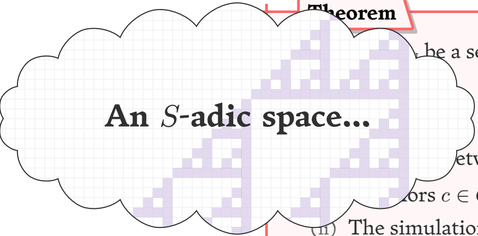

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Soficity: information $o(N^{d-1})$ in cubes $[\![N]\!]^d$

Example 1: seas of squares

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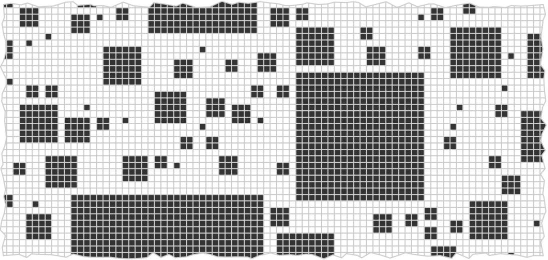

Seas of squares: [Westrick, 2017]

For $S \subseteq \mathbb{N}$ a Π_1^0-computable set, X draws independent ■ squares of sizes in S over a □ background.

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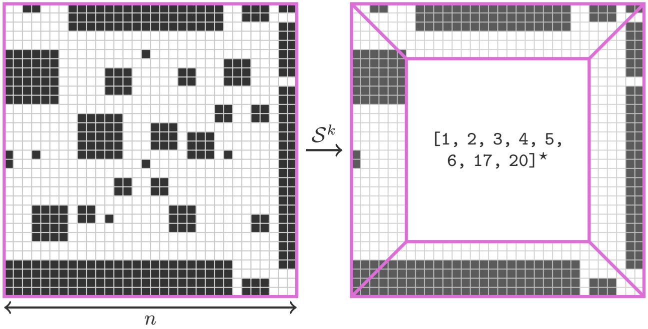

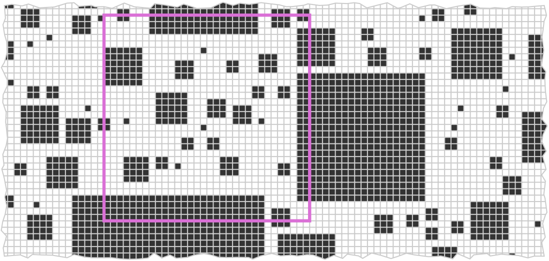

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colors of size $O(n^{2/3} \cdot \log n)$.

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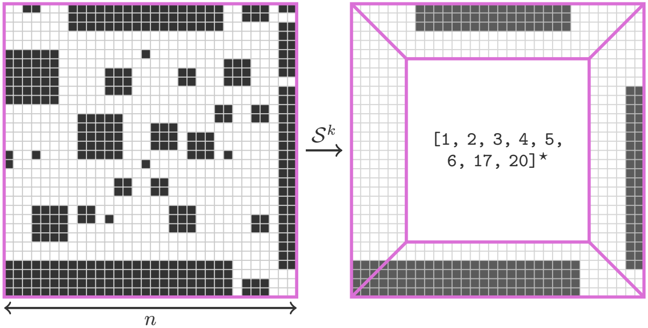

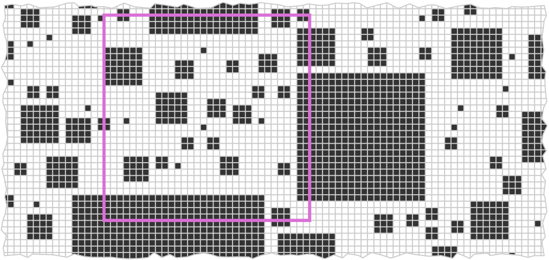

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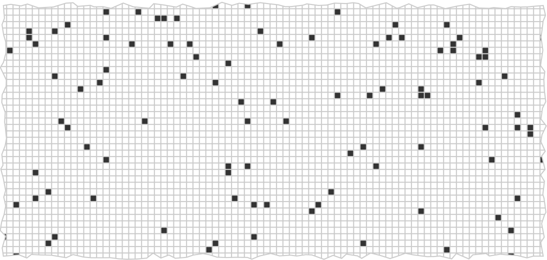

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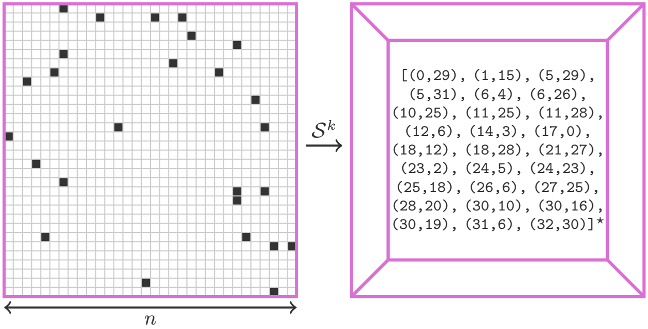

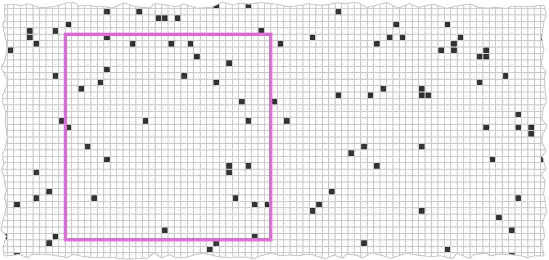

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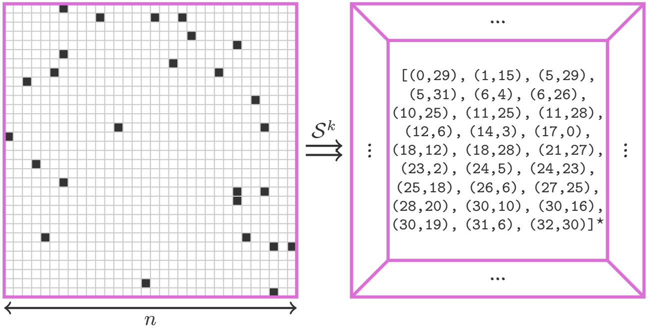

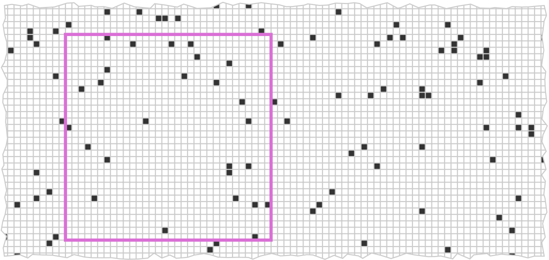

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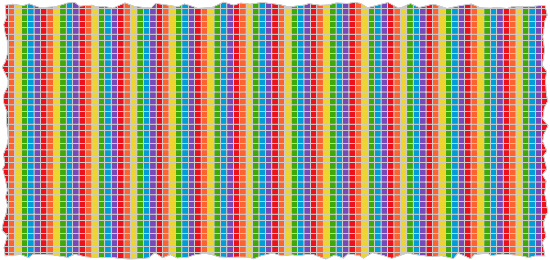
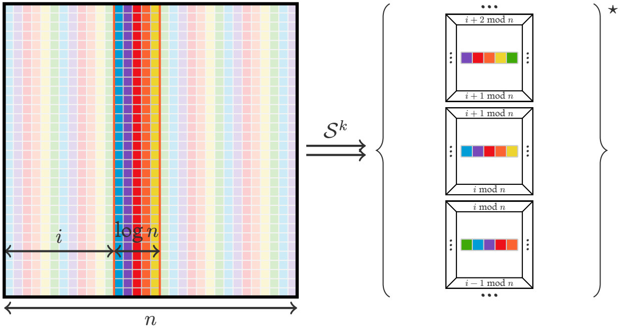


Periodic extensions: [DRS, 2010]

Given $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ effective, the configurations of $\mathcal{A}^{\mathbb{Z}^{d+1}}$ repeating some facet $x \in X$ vertically.

Example 3: periodic extensions



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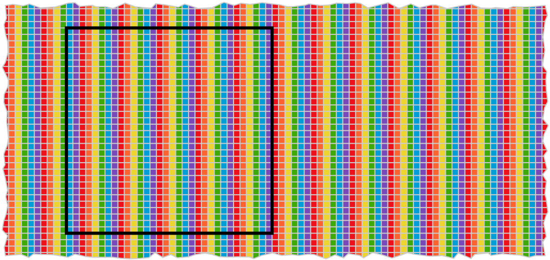


Periodic extensions: [DRS, 2010]

Given $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ effective, the configurations of $\mathcal{A}^{\mathbb{Z}^{d+1}}$ repeating some facet $x \in X$ vertically.

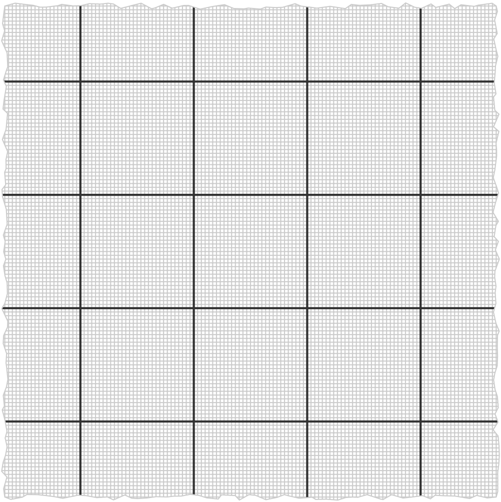
For a pattern of domain $[\![n]\!]^d$:
colors of size $O(\log n)$.

Proof outline

The fixpoint construction:

Define a sequence of tilesets $(T_\ell)_\ell$ as follows:

Proof outline



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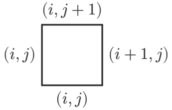
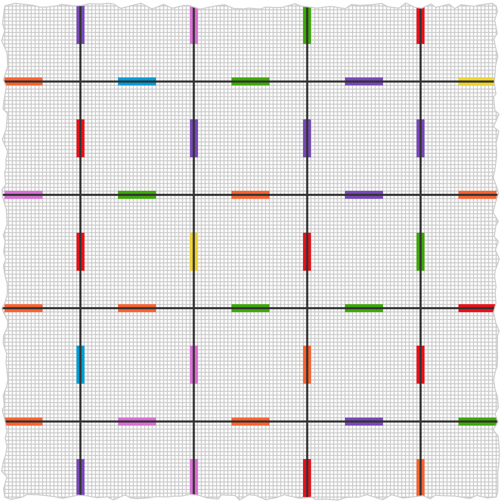
Define a sequence of tilesets $(T_\ell)_\ell$ as follows:

1. Encode the position of each tile in a block $[\![N_\ell]\!]^d$



Proof outline



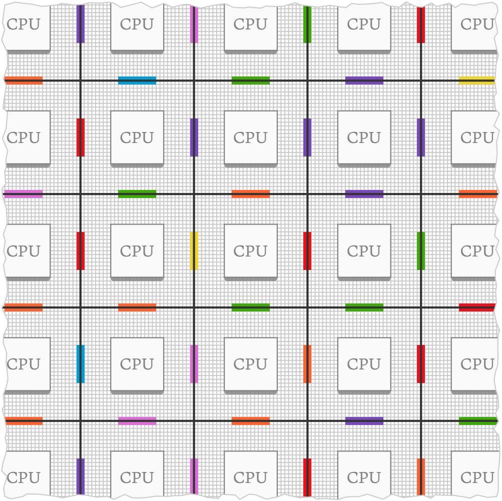
The fixpoint construction:

Define a sequence of tilesets $(T_\ell)_\ell$ as follows:

1. Encode the position of each tile in a block $[\![N_\ell]\!]^d$
2. Some border tiles contain a bit $b \in \{0,1\}$: their concatenation forms a *macro-color* $c \in \{0,1\}^*$;

If $i = N_\ell - 1$ and $j \simeq N_\ell/2$:

Proof outline



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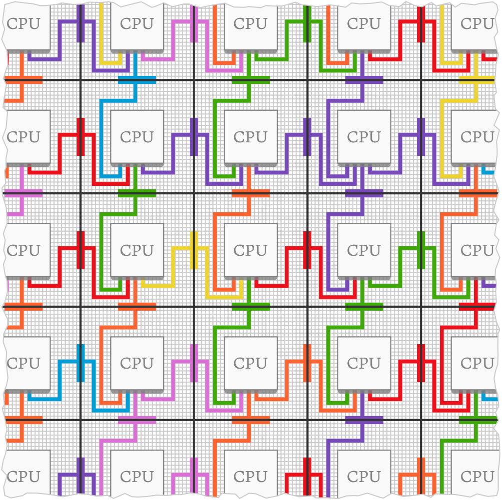
3. Embed computations to “program” $T_{\ell+1}$;
   role: check that a tuple of macro-colors emulates a tile of $T_{\ell+1}$

Proof outline



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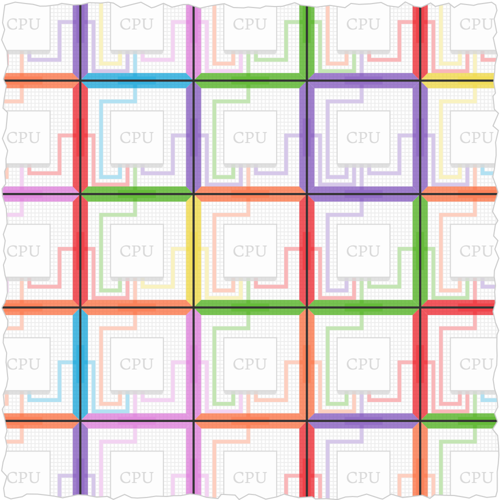

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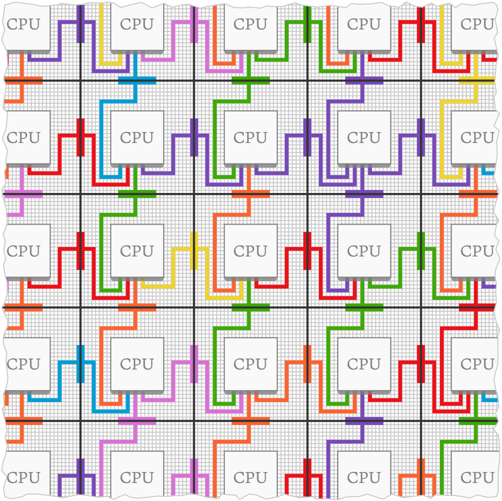

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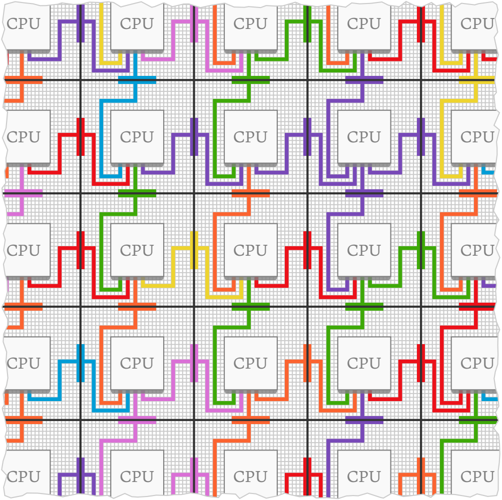

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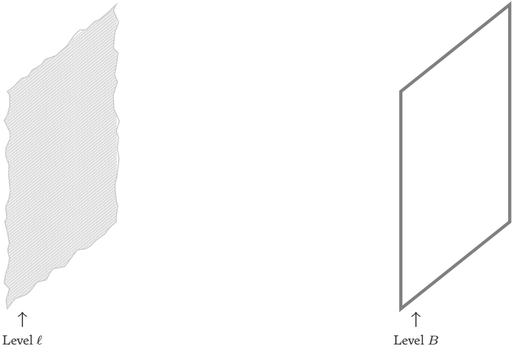
3. Embed computations to "program" $T_{\ell+1}$;
   role: check that a tuple of macro-colors emulates a tile of $T_{\ell+1}$
4. The programming of $T_{\ell+1}$ is required to build T_ℓ: solve this dependency by a fixed point argument!
   🚨 This only works if computations have enough room to terminate properly...

Proof outline (more details)



New simulation hierarchy:

Assume we are given some level ℓ:

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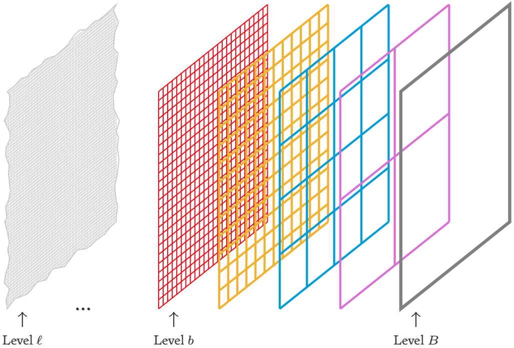

New simulation hierarchy:

Assume we are given some level ℓ:

1. Each T_ℓ-tile is involved in *many* macro-tiles (levels $b \leq j < B$ for some $b > \ell$);

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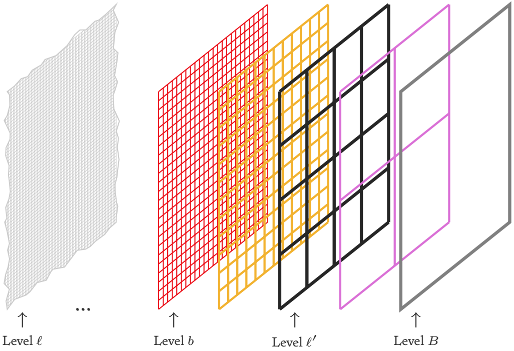

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1. Each T_ℓ-tile is involved in *many* macro-tiles (levels $b \leq j < B$ for some $b > \ell$);
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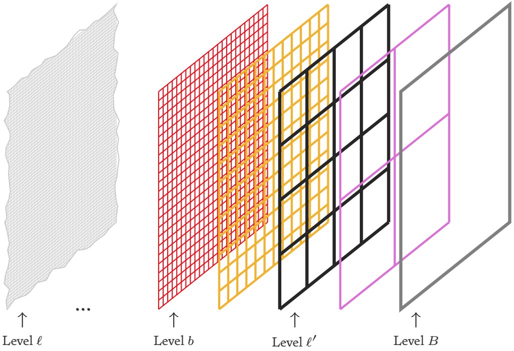

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2. Tiles in T_ℓ "*pick*" the next level of simulation ℓ' (such that $b \leq \ell' < B$);
3. Macro-tiles resp. embed the simulation steps $\mathcal{S}_j$ (for $b \leq j < B$).

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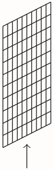
Drawing computation embeddings

Question

How many computation steps can one embed in a cube $[\![N]\!]^d$?

Proposition (Cube wiring theorem)



An $[\![N]\!]^{d-1}$ array of integers

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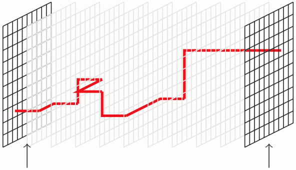

For any $[\![N]\!]^{d-1}$ array, there exists a wiring of depth N and $O(1)$ crossings sorting the cube.

Proof sketch [Thompson-Kung, 1976]. Sort rows (alternating order) and columns $\log N$ times:

47	46	65	56	12	66	41	49	61	2
57	60	55	19	52	13	89	17	94	97
42	91	85	74	10	4	11	77	44	67
82	73	18	86	20	87	23	38	90	0
78	1	35	43	29	7	37	84	14	5
76	39	54	22	21	9	15	48	51	26
79	72	40	96	6	24	75	3	27	80
58	31	45	98	50	25	83	59	68	53
70	28	69	33	64	8	93	32	99	16
30	88	92	36	63	71	34	81	95	62

□

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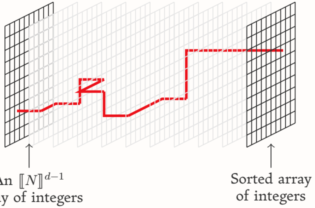

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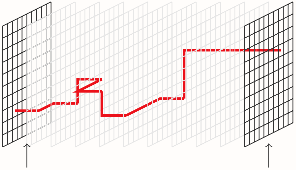

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57	60	55	19	52	13	89	17	94	97
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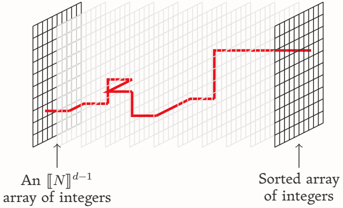

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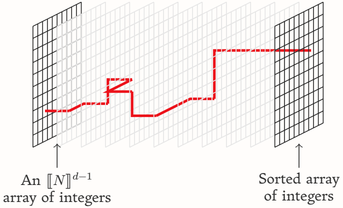

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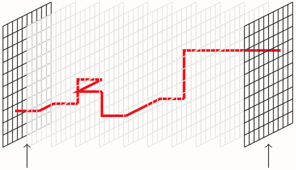

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4	10	11	42	44	67	74	77	85	91
90	87	86	82	73	38	23	20	18	0
1	5	7	14	29	35	37	43	78	84
76	54	51	48	39	26	22	21	15	9
3	6	24	27	40	72	75	79	80	96
98	83	68	59	58	53	50	45	31	25
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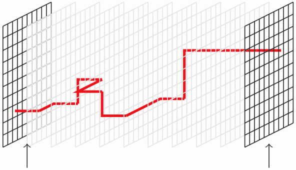

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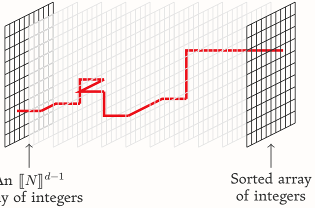

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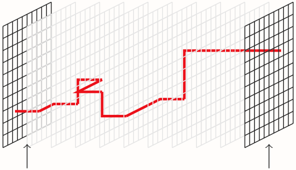

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1	5	7	14	29	26	22	19	15	0
2	6	11	27	33	35	23	20	17	9
3	10	24	32	39	38	37	21	18	13
4	12	28	42	40	49	50	36	31	25
8	16	41	46	44	53	52	43	34	30
76	54	51	48	47	55	56	45	65	66
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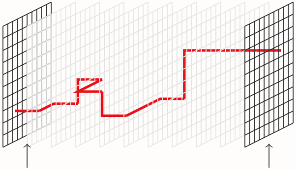

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0	1	5	7	14	15	19	22	26	29
35	33	27	23	20	17	11	9	6	2
3	10	13	18	21	24	32	37	38	39
50	49	42	40	36	31	28	25	12	4
8	16	30	34	41	43	44	46	52	53
76	66	65	56	55	54	51	48	47	45
57	59	61	62	63	68	78	83	84	90
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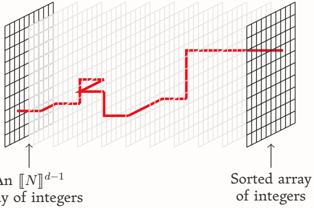

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35	33	30	34	36	31	32	37	38	39
50	49	42	40	41	43	44	46	47	45
57	59	61	56	55	54	51	48	52	53
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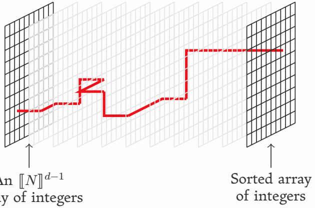

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8	16	21	23	24	25	26	27	28	29
39	38	37	36	35	34	33	32	31	30
40	41	42	43	44	45	46	47	49	50
61	59	57	56	55	54	53	52	51	48
58	60	62	63	64	65	66	67	68	69
80	78	77	76	75	74	73	72	71	70
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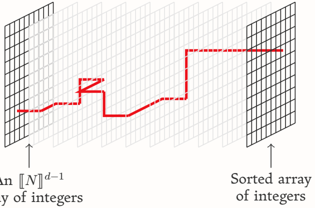

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39	38	37	36	35	34	33	32	31	30
40	41	42	43	44	45	46	47	49	48
58	59	57	56	55	54	53	52	51	50
61	60	62	63	64	65	66	67	68	69
79	78	77	76	75	74	73	72	71	70
80	81	82	83	84	86	87	89	88	85
99	98	97	96	94	93	92	90	91	95

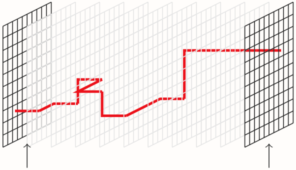
□

Drawing computation embeddings

Question

How many computation steps can one embed in a cube $[\![N]\!]^d$?

Proposition (Cube wiring theorem)



For any $[\![N]\!]^{d-1}$ array, there exists a wiring of depth N and $O(1)$ crossings sorting the cube.

Proof sketch [Thompson-Kung, 1976].
Sort rows (alternating order) and columns $\log N$ times:

0	1	2	3	4	5	6	7	9	10
19	18	17	16	15	14	13	12	11	8
20	21	22	23	24	25	26	27	28	29
39	38	37	36	35	34	33	32	31	30
40	41	42	43	44	45	46	47	48	49
59	58	57	56	55	54	53	52	51	50
60	61	62	63	64	65	66	67	68	69
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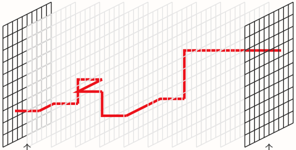

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Drawing computation embeddings

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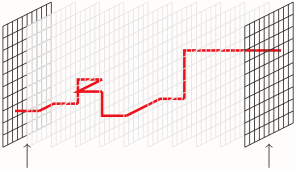

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80	81	82	83	84	85	86	87	88	89
99	98	97	96	95	94	93	92	91	90

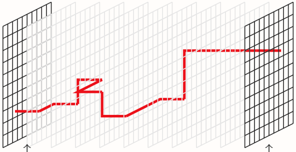
□

Drawing computation embeddings

Question

How many computation steps can one embed in a cube $[\![N]\!]^d$? $\implies$ $O(N^{d-1})$.

Proposition (Cube wiring theorem)



For any $[\![N]\!]^{d-1}$ array, there exists a wiring of depth N and $O(1)$ crossings sorting the cube.

Proof sketch [Thompson-Kung, 1976].
Sort rows (alternating order) and columns log N times:

0	1	2	3	4	5	6	7	8	9
19	18	17	16	15	14	13	12	11	10
20	21	22	23	24	25	26	27	28	29
39	38	37	36	35	34	33	32	31	30
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80	81	82	83	84	85	86	87	88	89
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□

RAM machines

- 32 variables $\mathtt{var}_0$, $\mathtt{var}_1$...
- An infinite memory $(\mathtt{M}[i])_{i\in\mathbb{N}}$.

Instructions: arithmetic operations on variables, and memory I/O.

Proof (folklore).
Given a trace of computation in chronological order,

$$\ldots(\mathtt{address}=6, \mathtt{time}=15, \mathtt{READ}, m)\,;\ (\mathtt{address}=42, \mathtt{time}=16, \mathtt{READ}, n)\ldots$$

check its correctness by sorting it lexicographically:

$$\ldots(\mathtt{address}=42, \mathtt{time}=3, \mathtt{WRITE}, n)\,;\ (\mathtt{address}=42, \mathtt{time}=16, \mathtt{READ}, n)\ldots$$ □

Conclusion

Sofic shifts generalize regular languages to infinite and higher-dimensional words.

In this talk

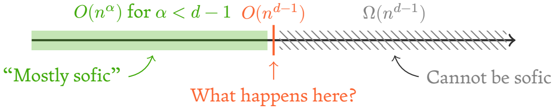
Proving soficity by quantifying the information in infinite sequences of substitutions.

Remarks:

- Block sizes anywhere in the interval $[\![2, 2^{O(L_\ell^\delta)}]\!]$;
- Generalizes to rectangular substitutions;
- Generalizes to non-uniform substitutions.

Questions:

- How much computations can be embedded in a rectangle $[\![n_1, \dots, n_d]\!]$?
- Generalization to other geometries (*i.e.* Cayley graphs of groups);
- Formalization of the $O(n^{d-1})$ argument.




That's all Folks!