Loria - Inria, Nancy (France)
ENS Paris-Saclay (France)

Descriptive complexity on non-Polish spaces

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Computability in Europe 2020

Loria
Laboratoire lorrain de recherche en informatique et ses applications

Inria

Introduction to represented spaces

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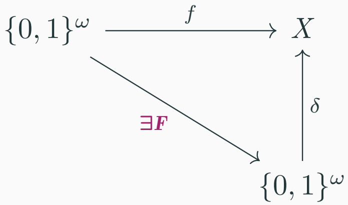

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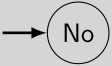

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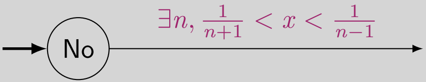

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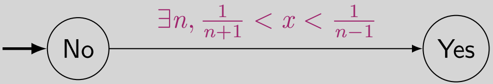

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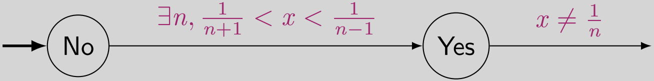

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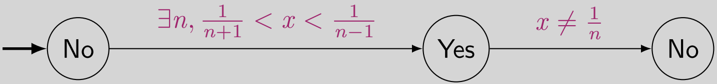

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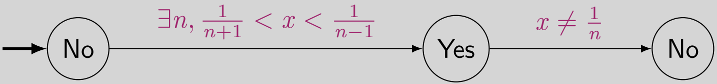

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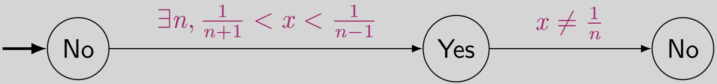

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Question:

Are those two approaches always equivalent?

Outline

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Formalization of the problem

Defining a notion of complexity

We need to build two notions of complexity for sets $S \subseteq X$:

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Key idea: semi-decidable problems are similar to (effective) open sets!

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- $[\Pi^0_1]$: problems P verifying $P^c \in [\Sigma^0_1]$, ie. :

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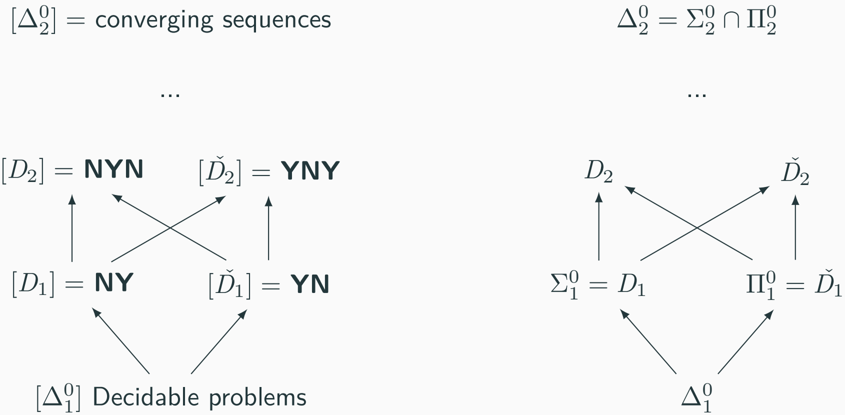
etc...
$\Delta^0_2 = \Sigma^0_2 \cap \Pi^0_2$,
where $\Sigma^0_2 = \{\bigcup_{n \in \mathbb{N}} U_n \setminus U'_n\}$.

Reformulation of the problem



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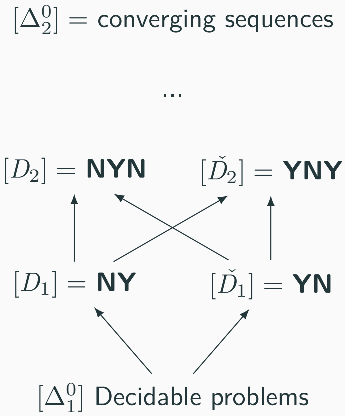

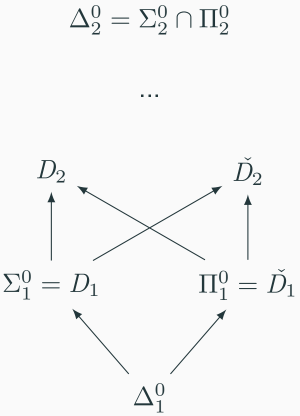

For $\Gamma \in \{D_\eta : \eta < \omega_1\}$, when does one have $\Gamma = [\Gamma]$?

Countably-based spaces

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Theorem 2.1: M. De Brecht, 2012

For (X, δ) a countably-based represented space,

$$S \in \Gamma \iff S \in [\Gamma]$$

Countably-based spaces: matching of the two hierarchies

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[New] - Theorem 2.2: Hierarchies coincide in an effective way

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in an effective and uniform way.

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In cb_0 spaces, if $\check{\Gamma} = \{P : P^c \in \Gamma\}$,then:

[New] - Theorem 2.7: Hardness-criterion

$$S \notin D_\eta \iff S \text{ is } \check{D}_\eta\text{-hard.}$$

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Theorem 2.8: Hardness-criterion

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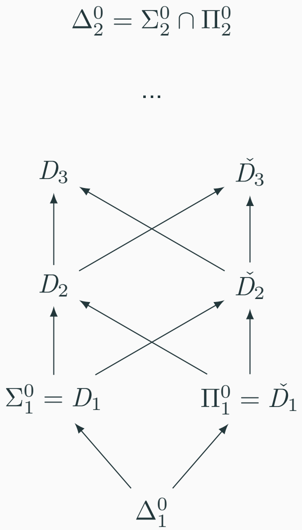

Non countably-based spaces

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It is an admissible representation of $\mathbb{R}[X]$.

Complexity of S_1

Property 3.1: Complexity of S_1 (Part 1)

$$S_1 = \left\{ P \in \mathbb{R}[X] : p_0 = 0 \text{ or } p_0 > \frac{1}{\deg(P)} \right\} \in [D_2]$$

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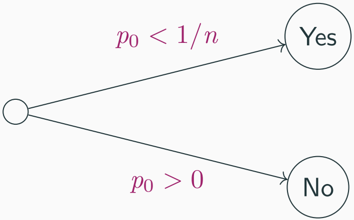

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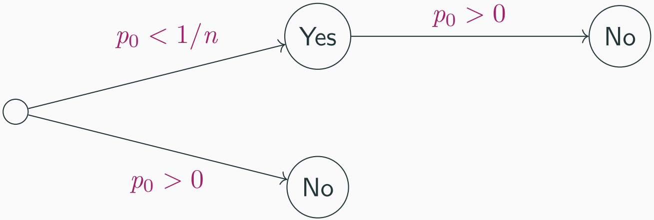

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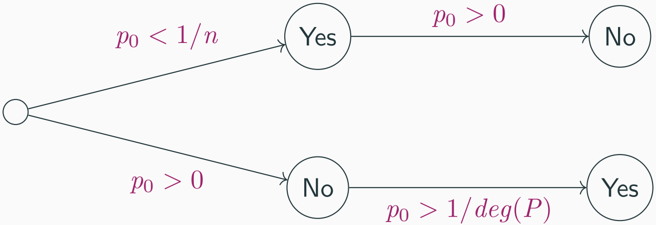

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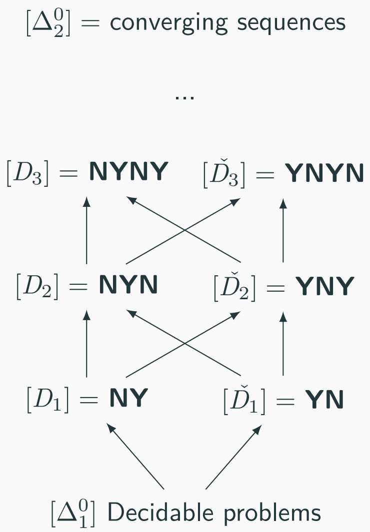
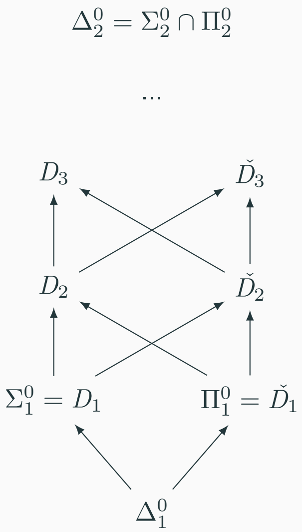
Conclusion : $S_1 \in [D_2]$ but $S_1 \notin D_2$

Complexity of S_1





To go even further... Introduce S_2.

$$S_2 = \left\{ \frac{1}{k_1} + \frac{X^{k_1}}{k_2} + ... + \frac{X^{k_n}}{k_{n+1}} : \exists n, k_1 < ... < k_{n+1}, k_n \text{ even} \right\}$$

[New] - Theorem 3.3: Complexity of S_2

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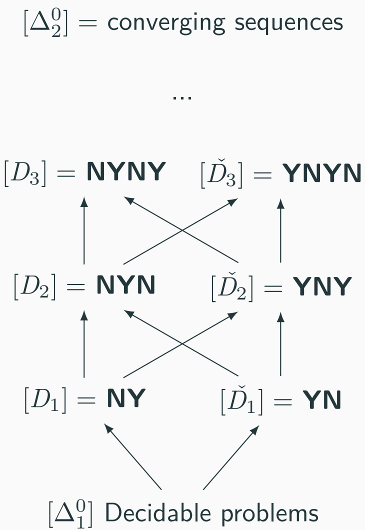
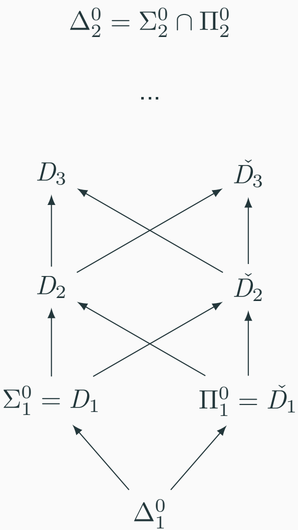
Complexity of S_2





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Guess: The difference is related to the mismatch between **topological** and **sequential** aspects.

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Why do we think that?

[New] - Theorem 3.4: $[\Gamma]$ depends on sequentiality

For a coPolish space X, $1.$ and $2.$ are equivalent.

1. Closure and sequential closure coincide on X.
2. For every $n < \omega$, $A \in [D_n]$ iff $A \in D_n$.

Consequences - About hardness

Recall that the definition of Γ-hardness involves $\{0,1\}^\omega$.

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For example:

[New] Theorem 3.5: Countably-based spaces

$$A \notin D_\eta \iff A \text{ is } \check{D}_\eta\text{-hard.}$$

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For example:

[New] Theorem 3.6: co-Polish spaces

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DST works better with symbolic complexity (and represented spaces)
rather than with topology.

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Let us recap:

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Open questions: Why is the equivalence broken? On which spaces does it happen, at which complexity levels, and why?

Our guess: The difference is related to the mismatch between **topological** and **sequential** aspects.

Questions?