Surface Entropies of $\mathbb{Z}^2$ Subshifts

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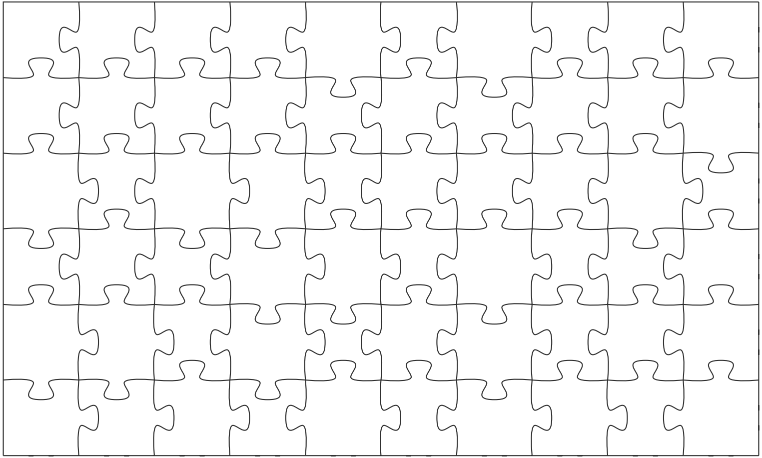
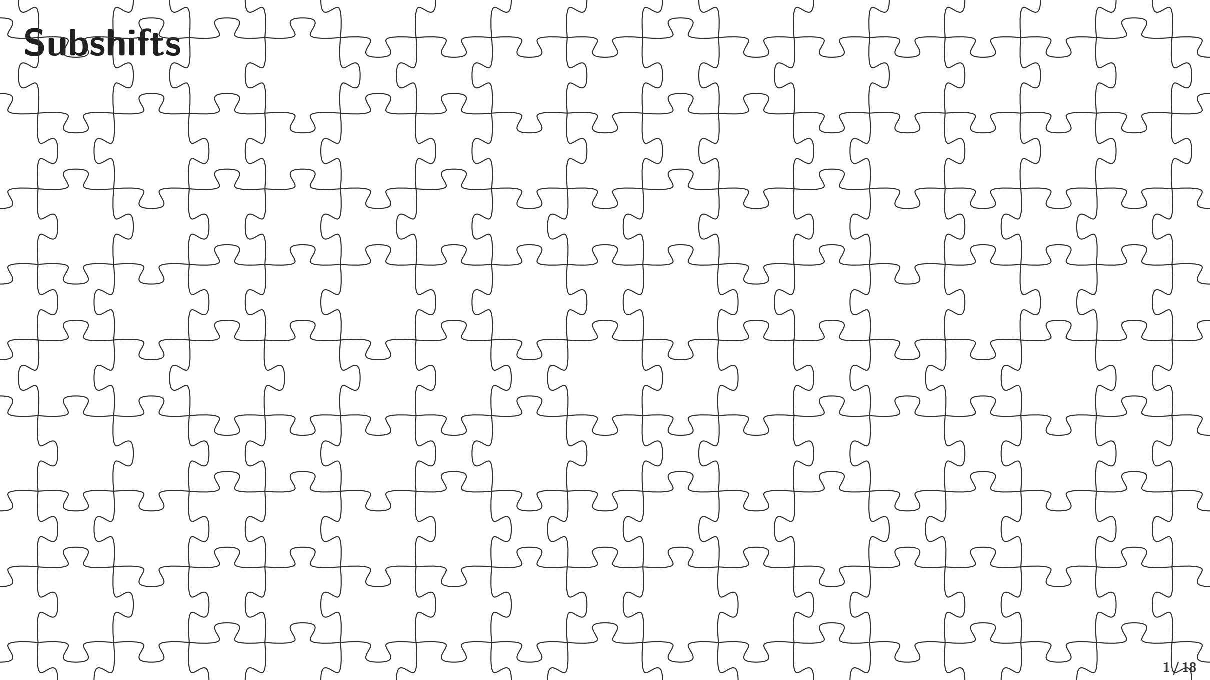
Subshifts



Subshifts



Subshifts

Definition 1 2D Shifts

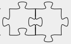
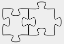
A 2D-*subshift* is a set of colorings $\mathbb{Z}^2 \mapsto \Sigma$ that do not contain some family of forbidden patterns $\mathcal{F}$. Each family of forbidden patterns defines a subshift:

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^2} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

Example 2 Jigsaw puzzle

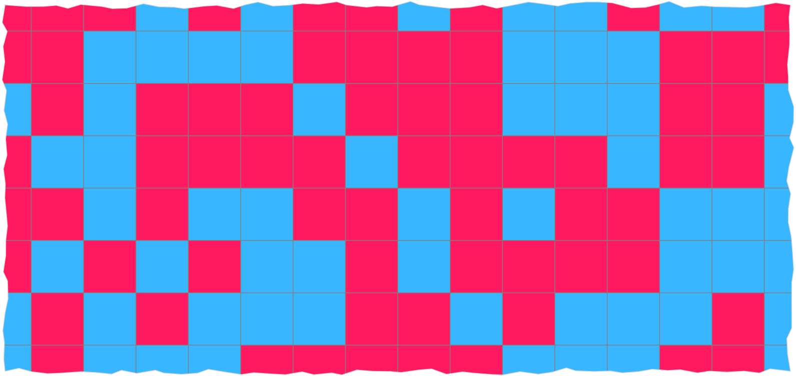
$\Sigma =$ , , etc...

and

$\mathcal{F} = \{$ , , etc... $\}$

Example

$$\mathcal{F} = \emptyset$$



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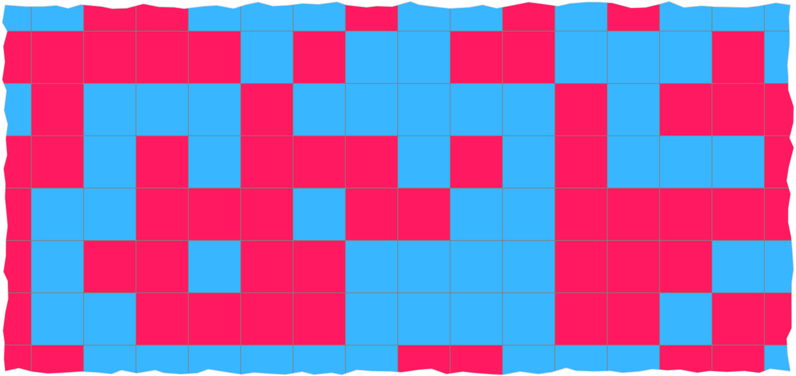

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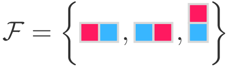

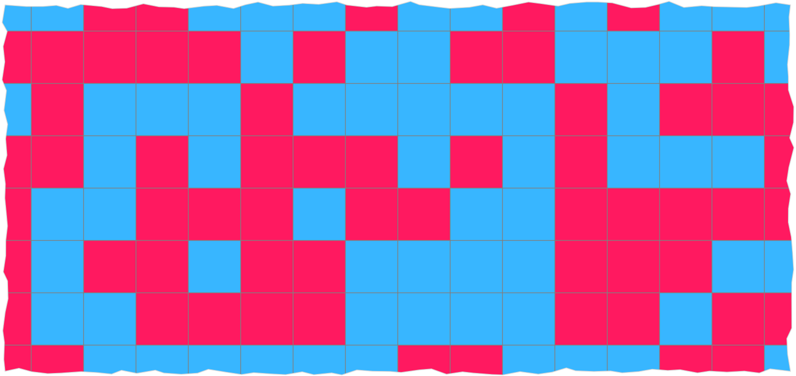

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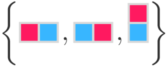
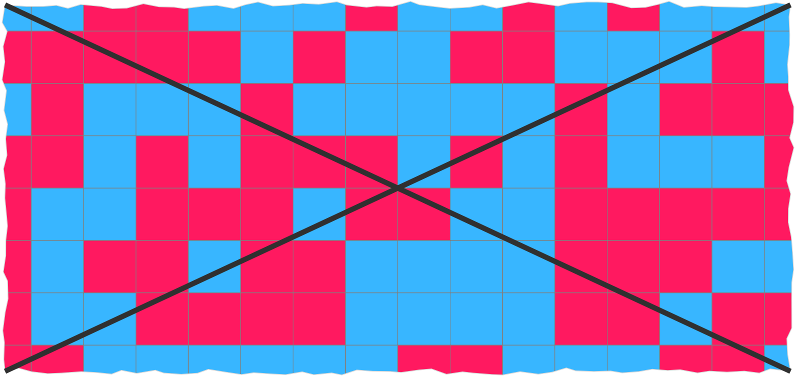
$\mathcal{F} =$ 



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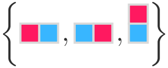
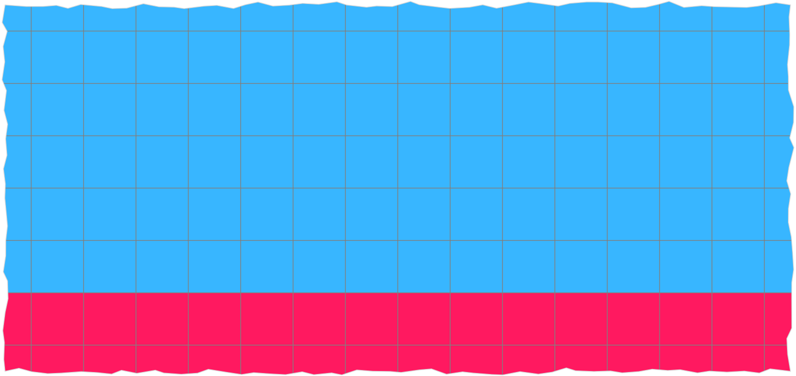
$\mathcal{F} =$ 



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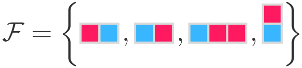

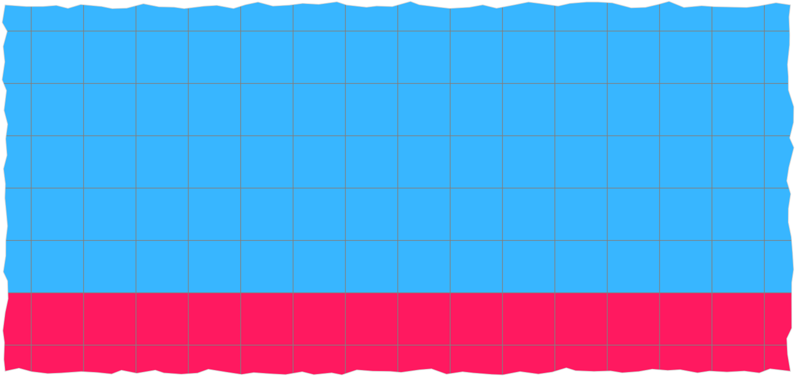

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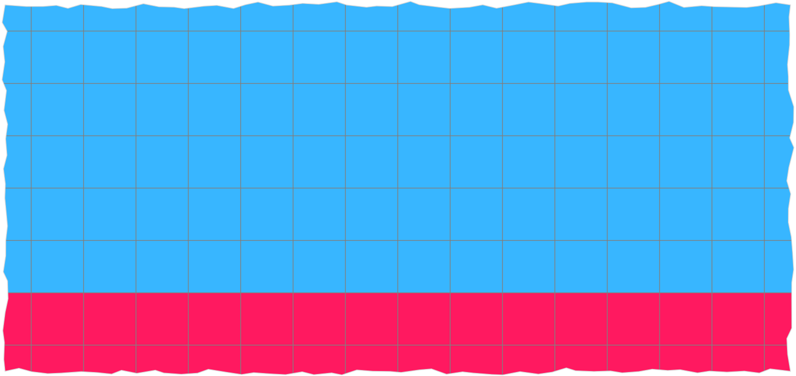

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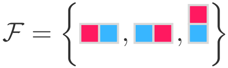

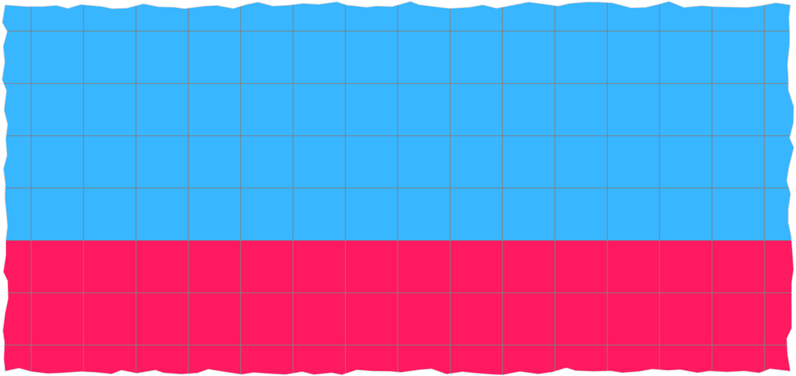

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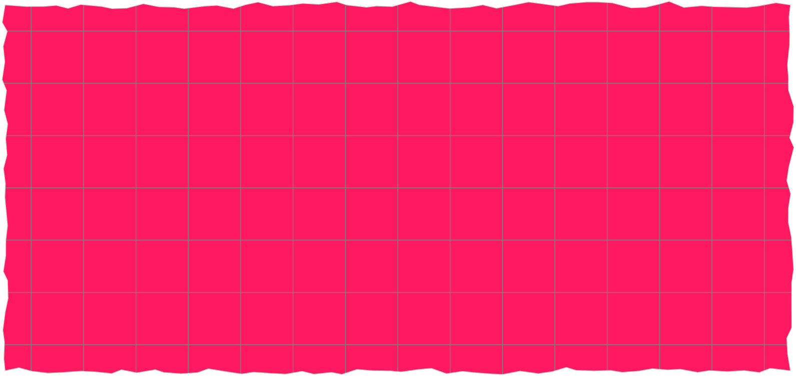

Subshifts as computation models

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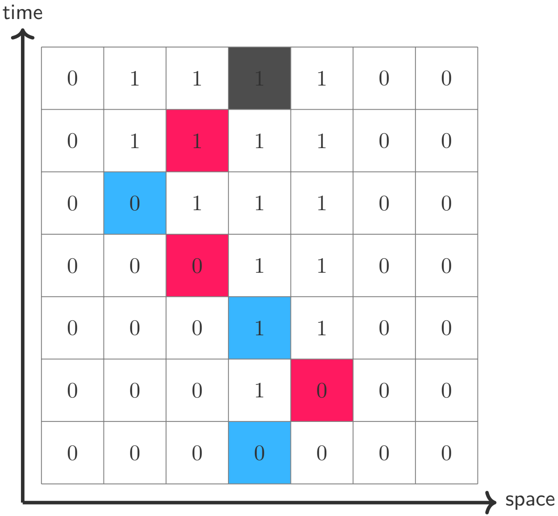

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0	1	1	1	1	0	0
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0	0	1	1	1	0	0
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Classes of subshifts

Definition 3 Classification of shifts

1. A *subshift of finite type* (or SFT) is a subshift that can be defined by a finite family of forbidden patterns.
2. An *effective subshift* is a subshift that can be defined by a recursively enumerable family of forbidden patterns.

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Theorem 4 [Hochman 2010, DRS 2012, AS 2013]

For any effective 1D subshift X_1, there exists a 2D SFT X_2 which *simulates* X_1.

Topological entropy

Complexity function

Definition 5 Complexity function

The *complexity function* $N_n(X)$ is defined as the number of different patterns of size $n \times n$ that appear in X.

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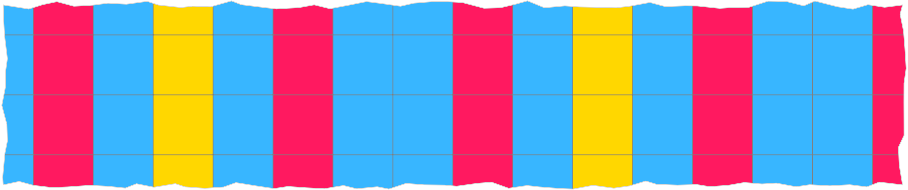

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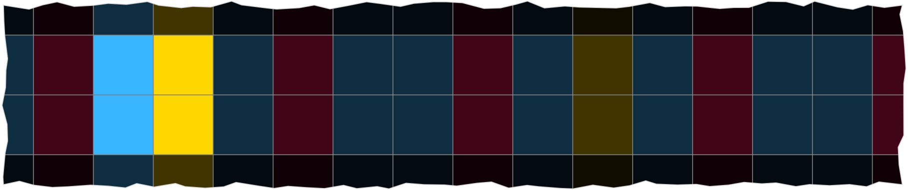

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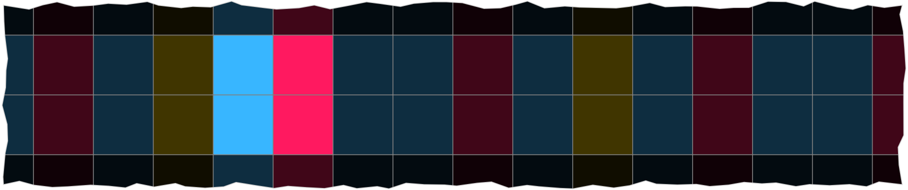

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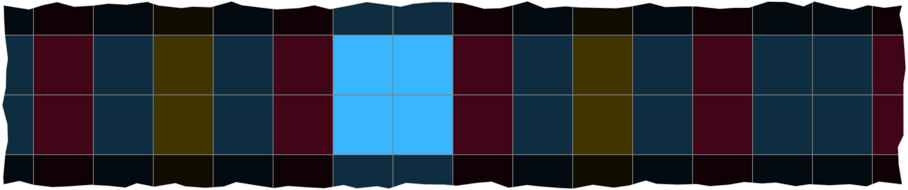

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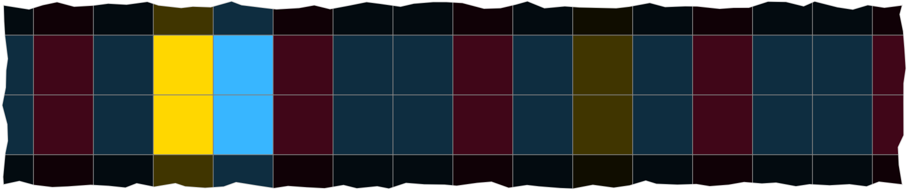

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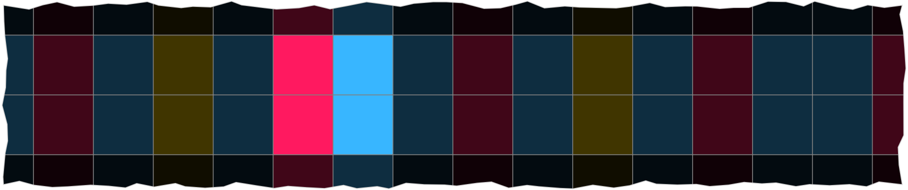

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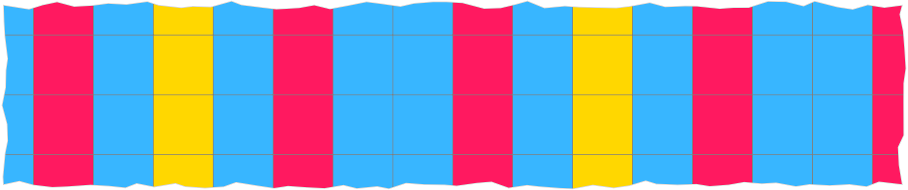

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Complexity function is density

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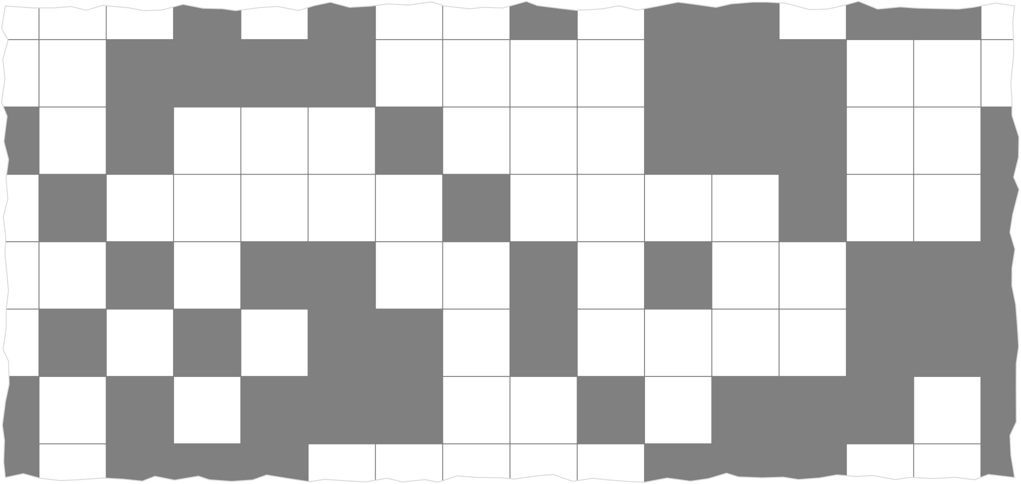

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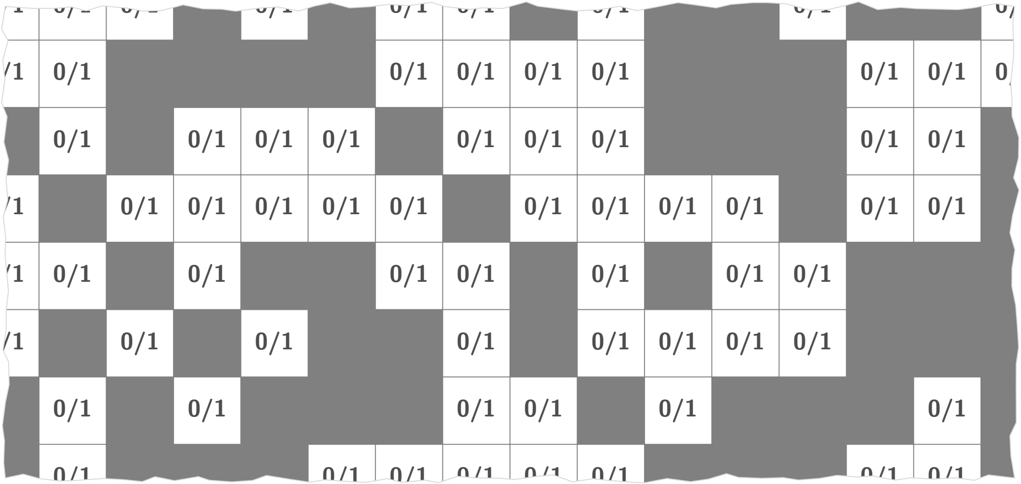

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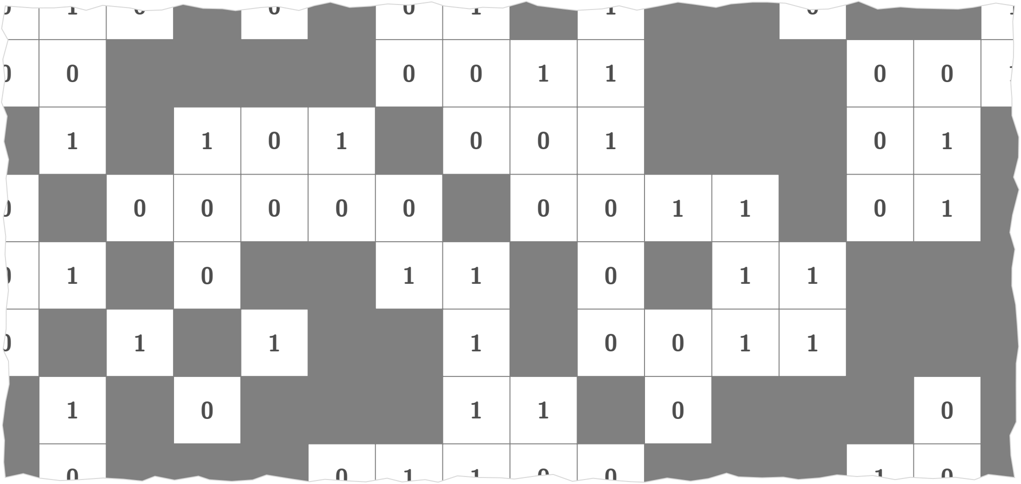

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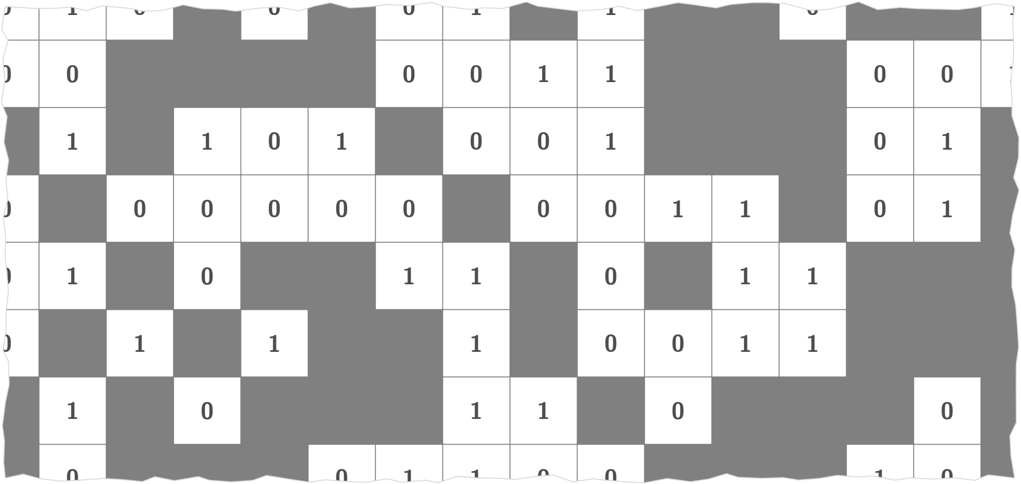

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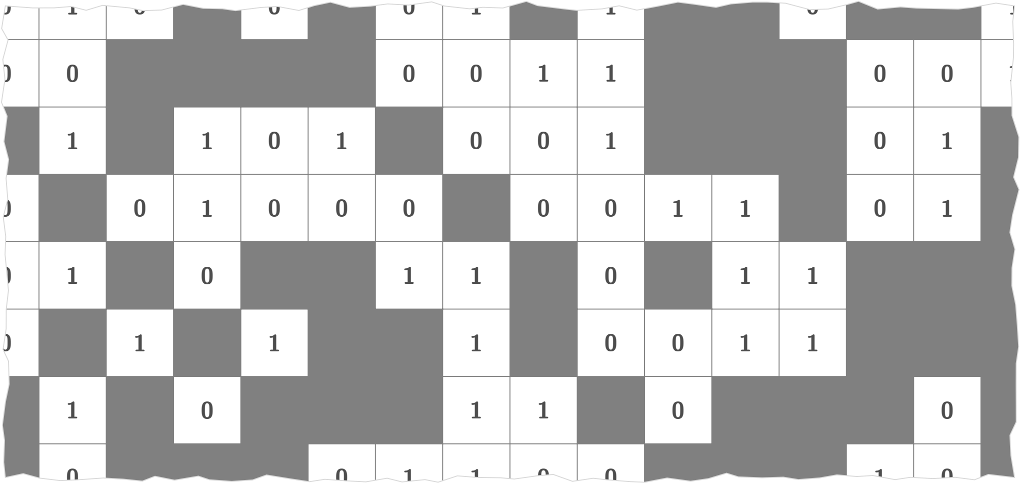

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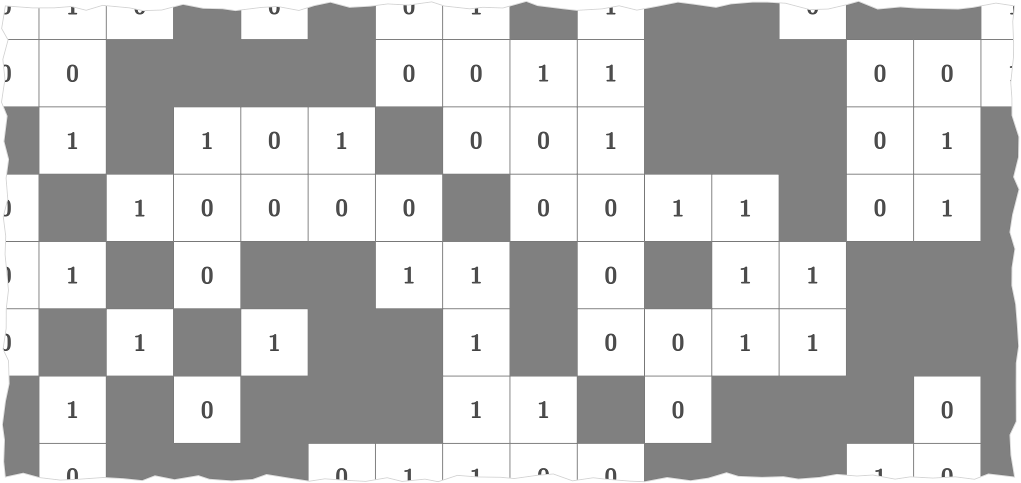

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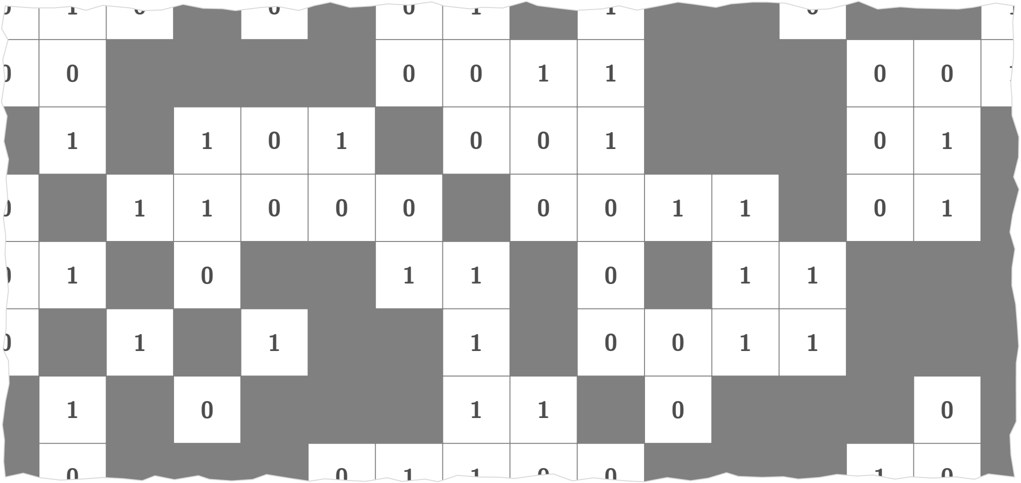

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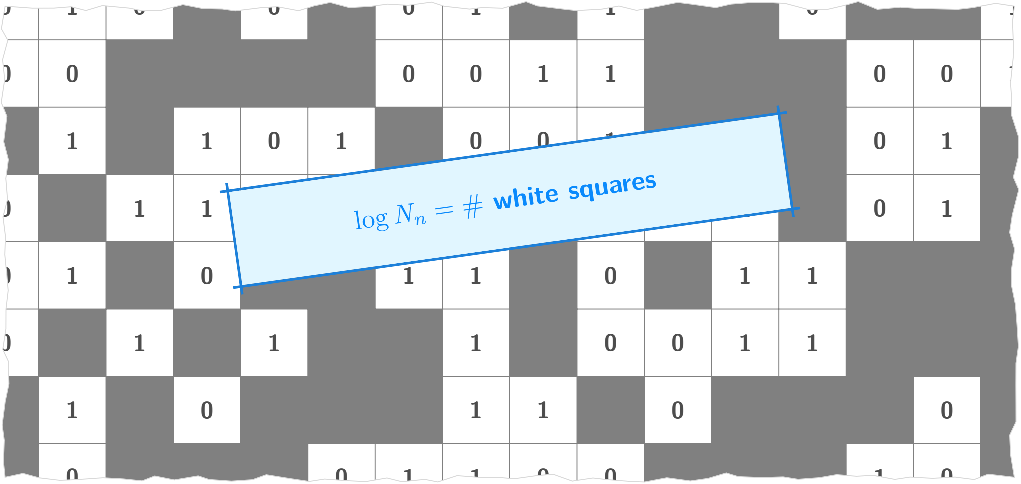

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QUESTION:

What are the possible values for $h_{top}(X)$ for all the SFTs?

What about topological entropies? (Part 1)

$$\log N_n \simeq h_{top}(X)n^2$$

[Hochman & Meyerovitch, 2010] proved that topological entropies were **exactly** the right-computable real numbers:

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$\Longleftarrow$ For any right-computable h, we create an SFT X such that $h_{top}(X) = h$.

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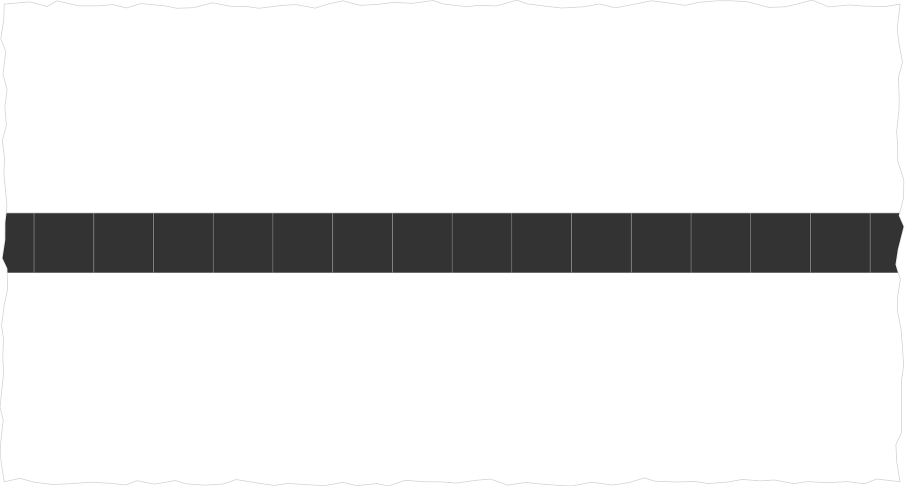

$h = .10100\ldots$

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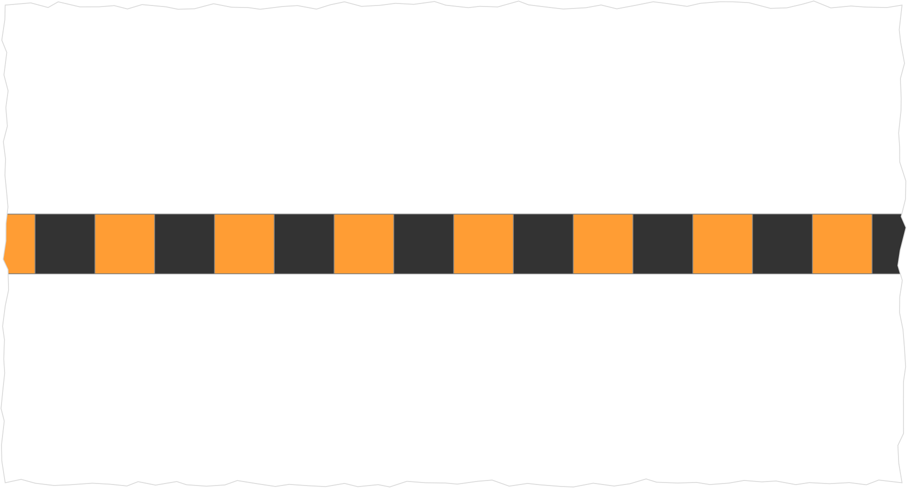

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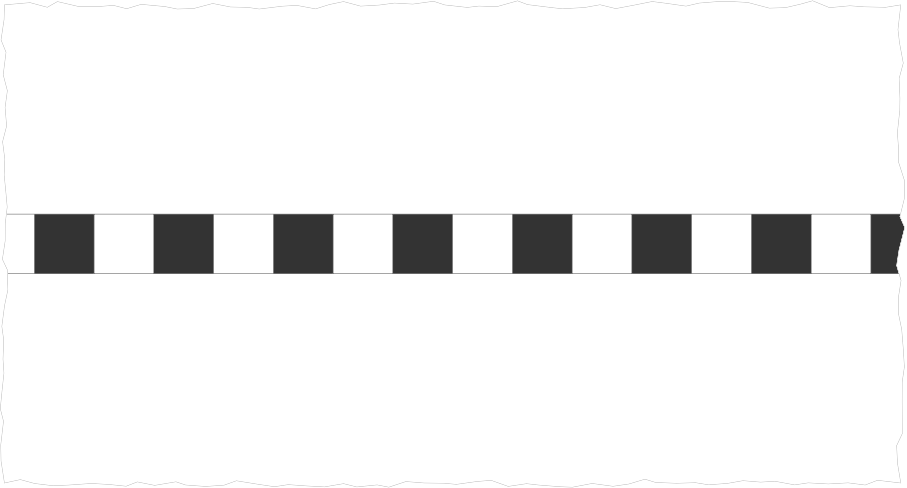

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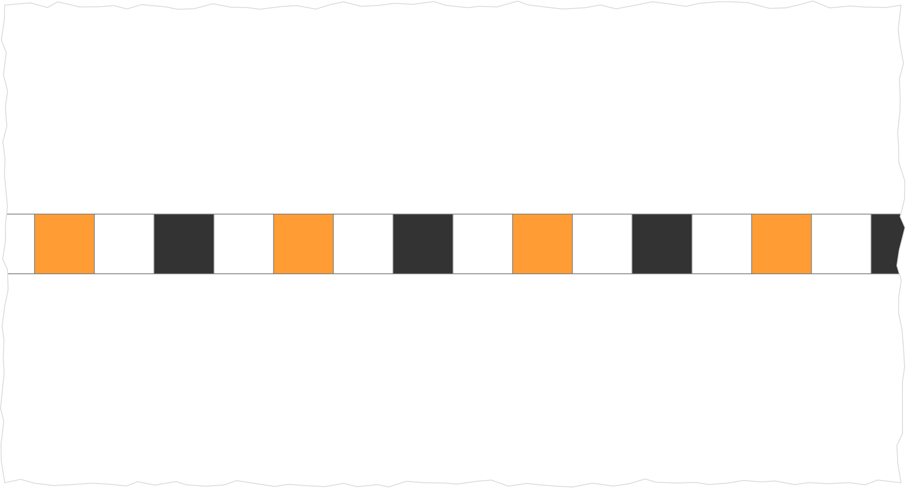

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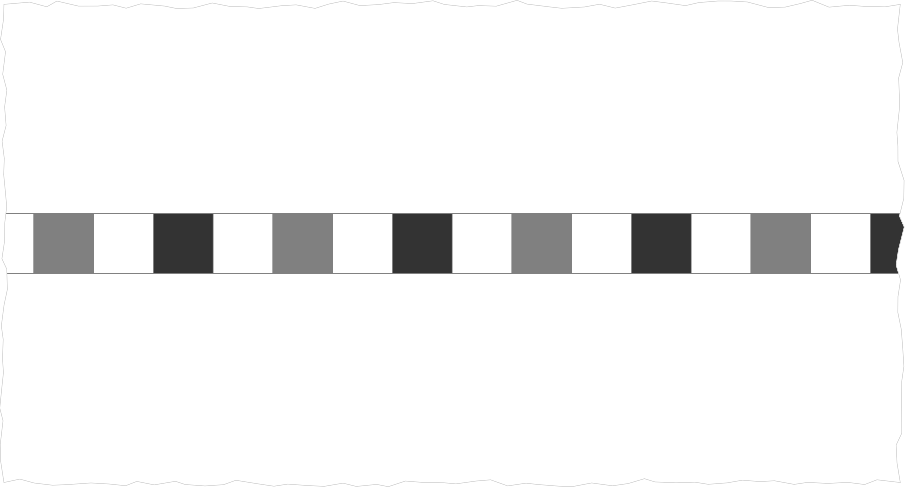

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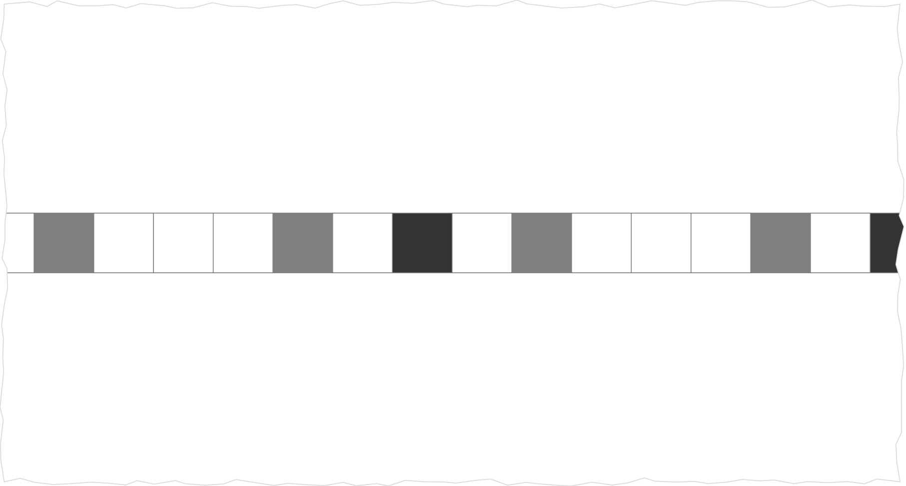

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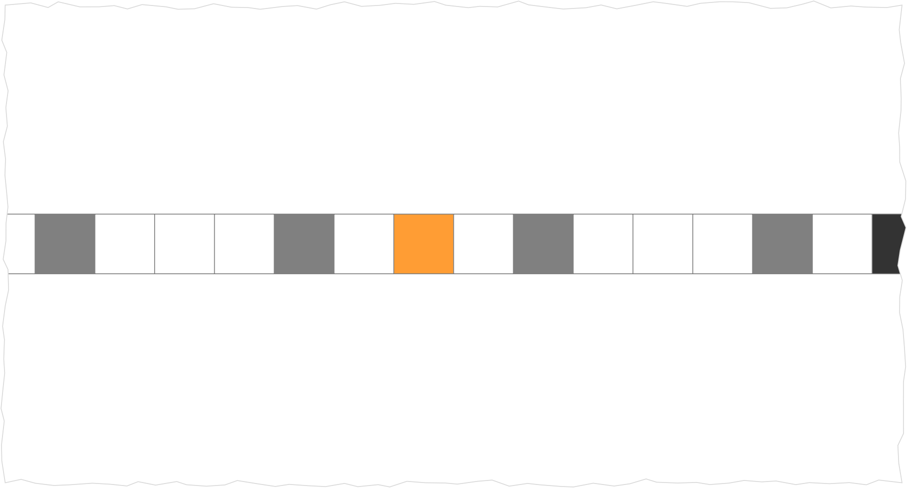

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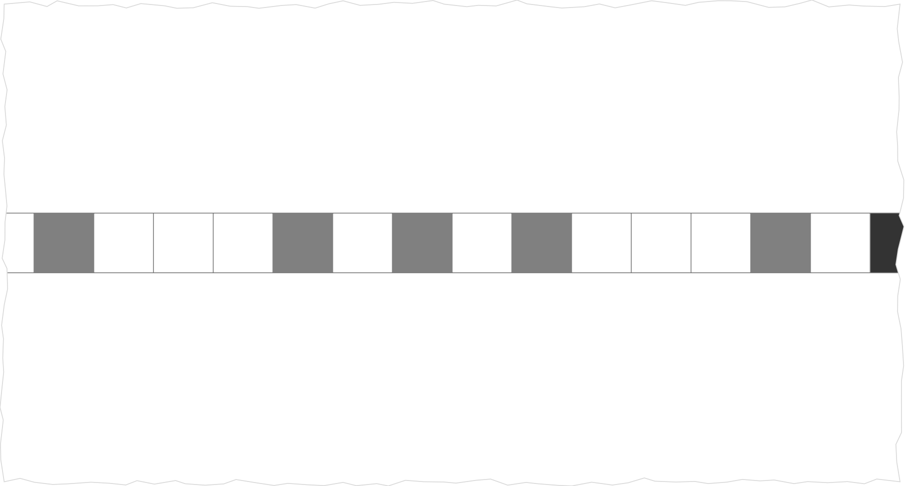

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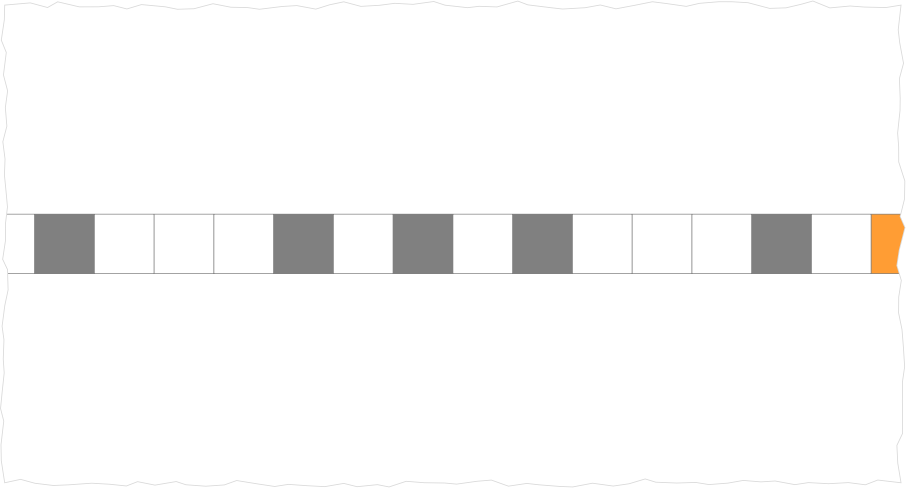

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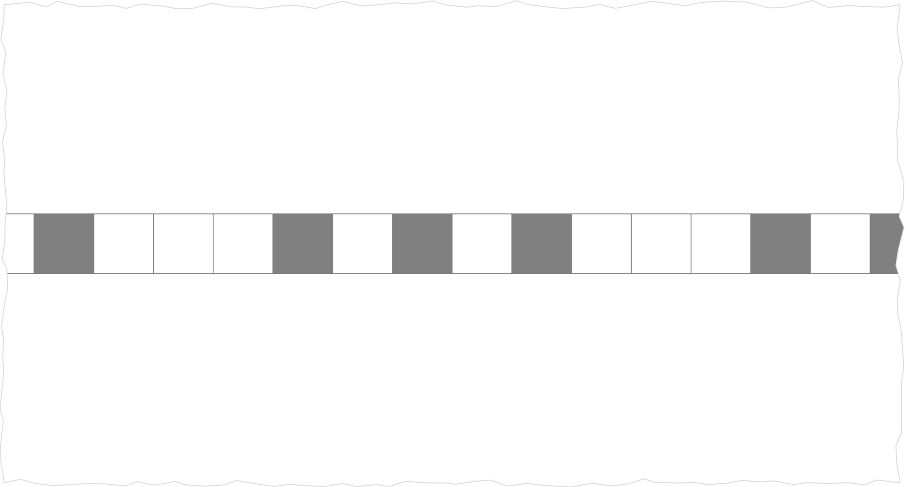

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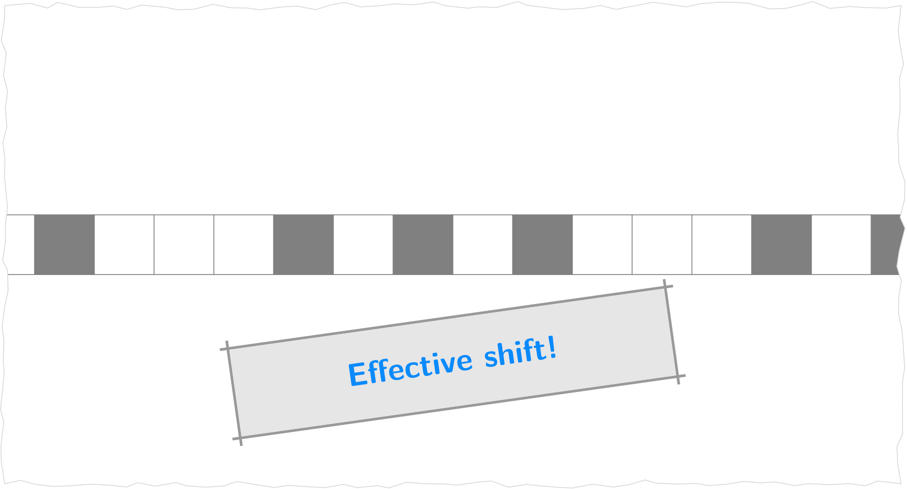

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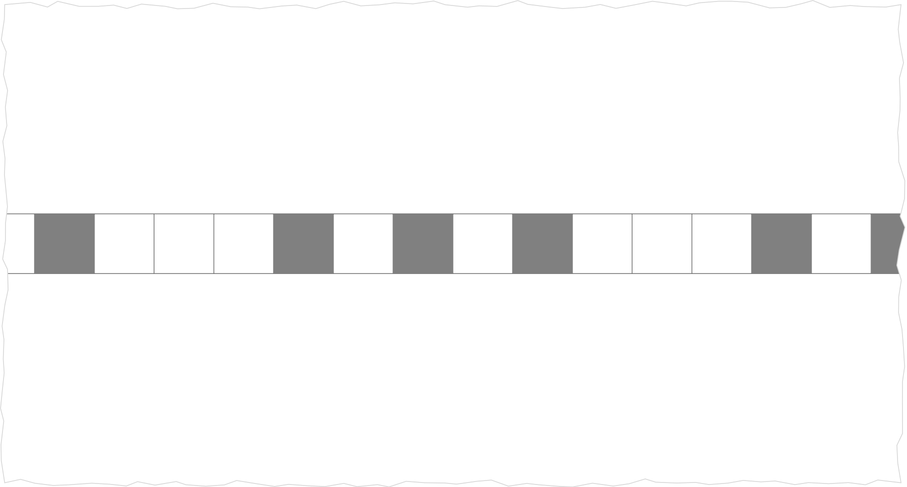

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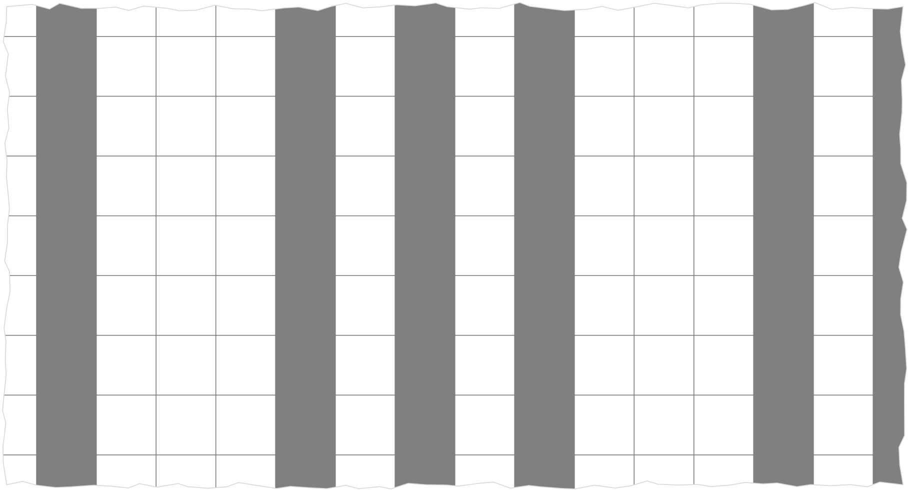

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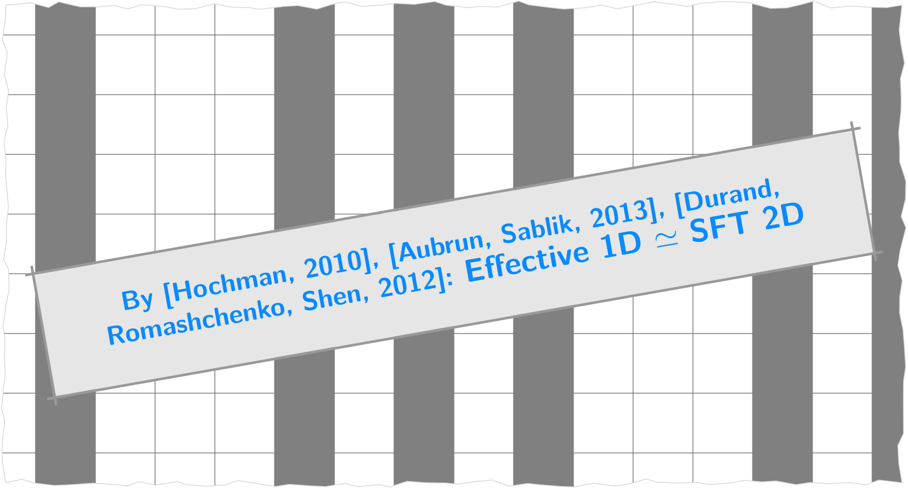

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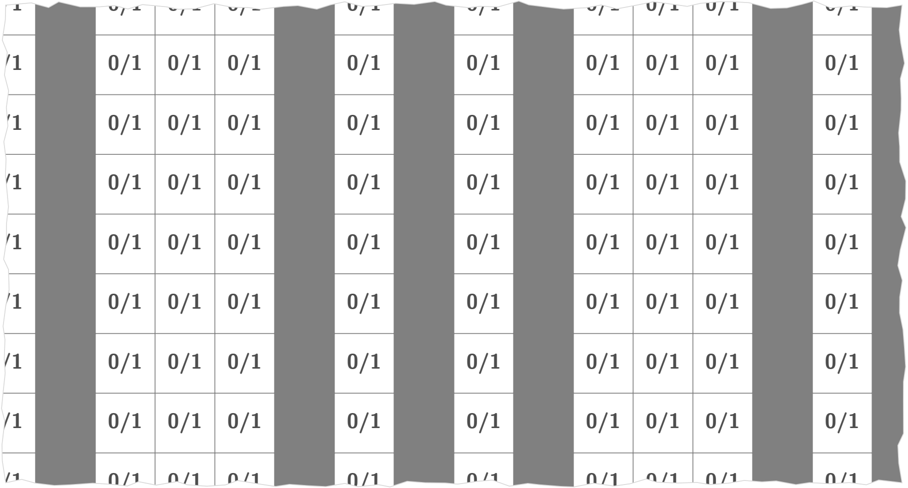

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	0	0	0		0		1		0	1	1		0
	0	0	0		1		1		0	1	1		0
	1	1	1		1		1		0	1	1		1
	0	0	0		0		1		0	0	0		0
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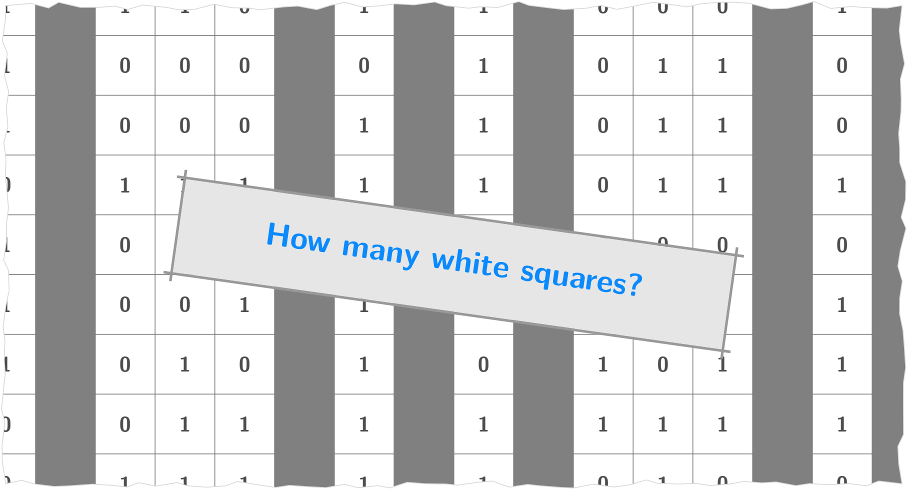

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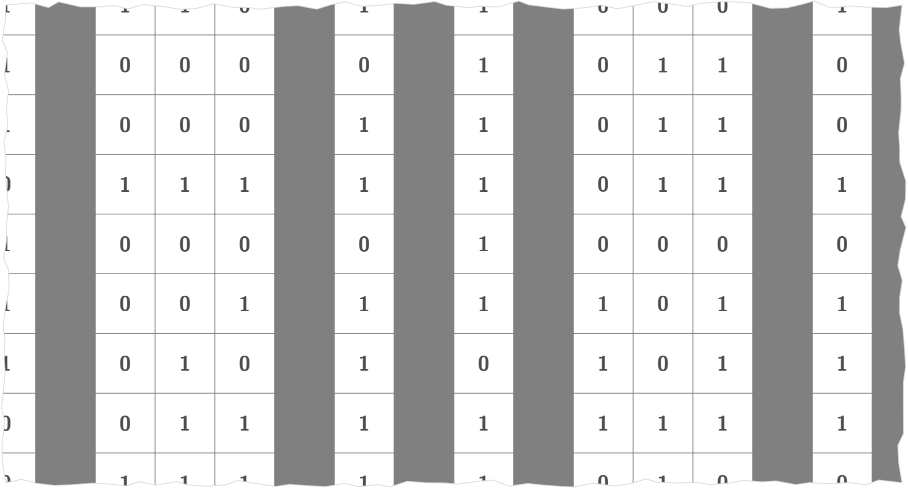

$h = .10100\ldots$

What about topological entropies? (Part 3)

$$\log N_n \simeq hn^2 \qquad \Longrightarrow \qquad h_{\mathsf{top}} = h$$

Surface entropy

The arithmetical hierarchy

Definition 7 Arithmetical hierarchy of real numbers

ARITHMETICAL HIERARCHY OF REAL NUMBERS

1.

$$\Delta_1 = \text{computable}$$
$$\Sigma_1 = \text{left-computable}$$
$$\Pi_1 = \text{right-computable}$$

2.

$$\Delta_2 = \text{limits of r.e. rationals}$$
$$\Sigma_2 = \text{supremum of right-computable}$$
$$\Pi_2 = \text{infimum of left-computable}$$

3. ...

What happens after the quadratic term?

We already said, for a 2D SFT:

$$\log N_n \simeq h_{top} n^2$$

Question:

Are there other possible asymptotic growths?

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Are there other possible asymptotic growths?

Yes! By [Meyerovitch, 2011], the class of α such that

$$\log N_n \simeq K n^{\alpha}$$

is exactly the class of Π_3 real numbers.

Surface entropies

The idea: linear term

$$\log N_n \simeq h' n$$

Surface entropies

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Definition 8 Surface entropy [Pace, 2018]

The *surface entropy* $h_s(X)$ of a shift X is defined as:

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Surface entropies

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Question:

What are the possible values for $h_s(X)$ for all the SFTs?

Surface entropies (Part 1)

$x \in \Pi_3$ if there exists a recursively enumerable $(r_k)_{k \in \mathbb{N}}$ in Π_1 such that:

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- $\frac{\log N_n(X)}{n^2} \in \Pi_1$;
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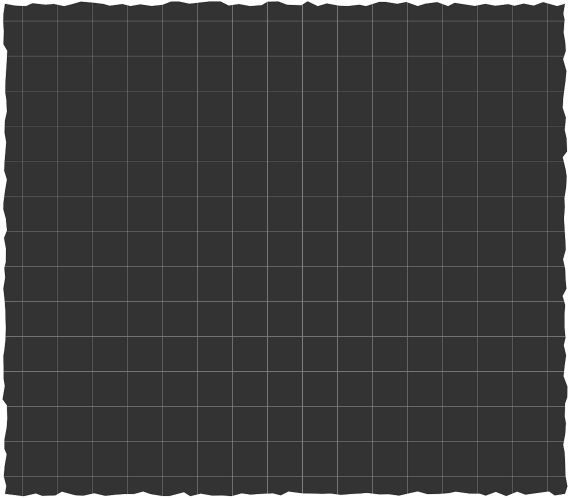
- $h_s(X) = \limsup \ldots = \inf \sup(\Pi_1) \in \Pi_3$;

$\Longleftarrow$ For any $h' \in \Pi_3$, we need to create an SFT X such that $h_s(X) = h'$.

Surface entropies: the sparse squares (Part 2)



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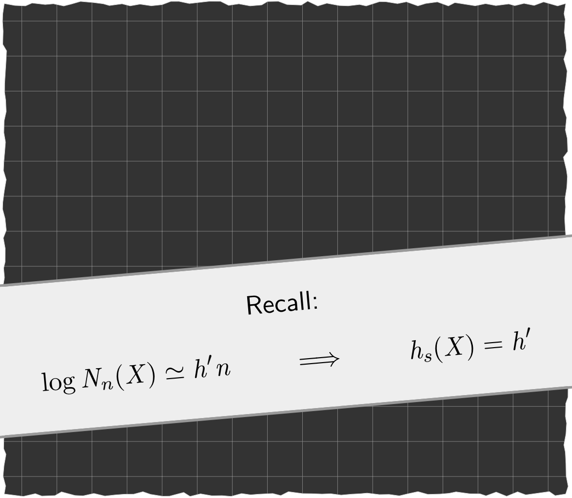
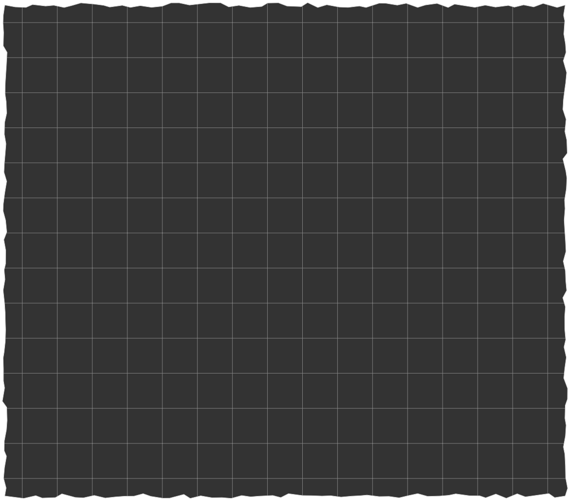


Recall:

$$\log N_n(X) \simeq h' n \qquad \Longrightarrow \qquad h_s(X) = h'$$

Surface entropies: the sparse squares (Part 2)



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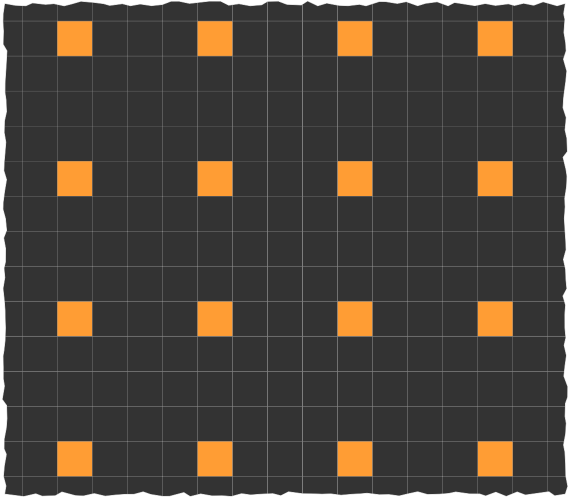

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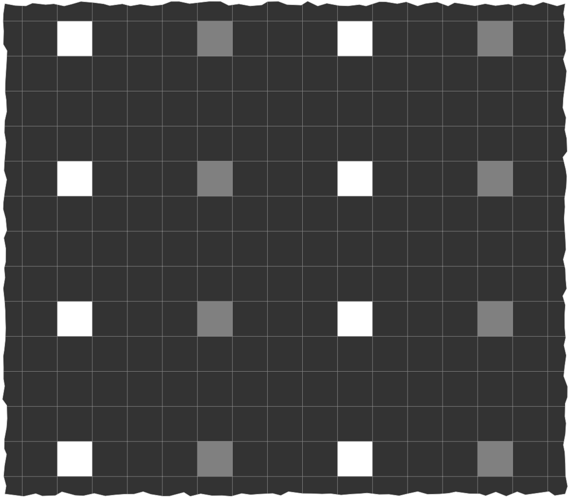
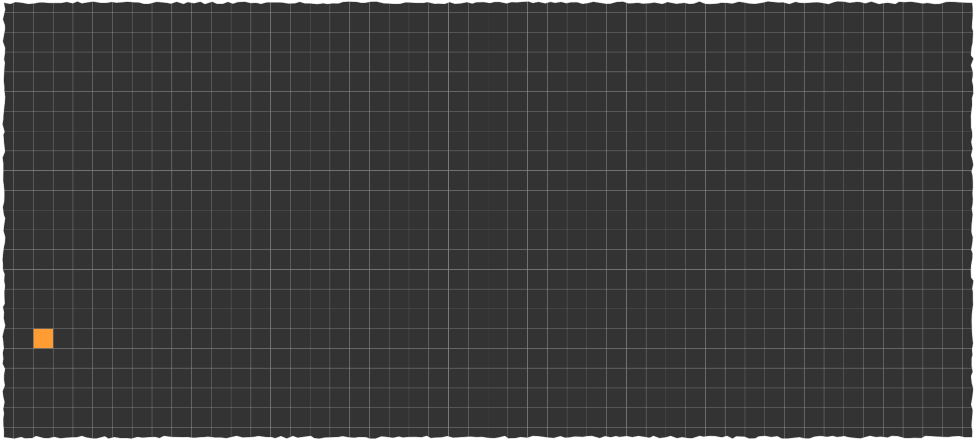


$r_4 = 1/2$

Surface entropies: structure (Part 3)



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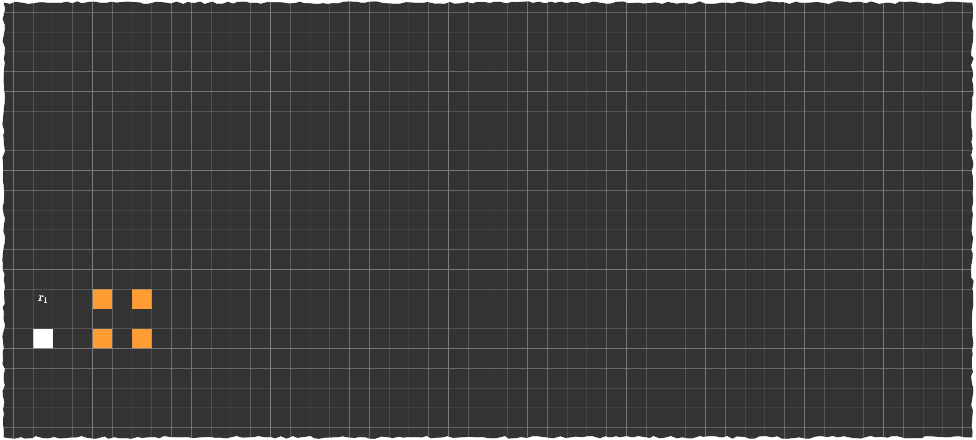

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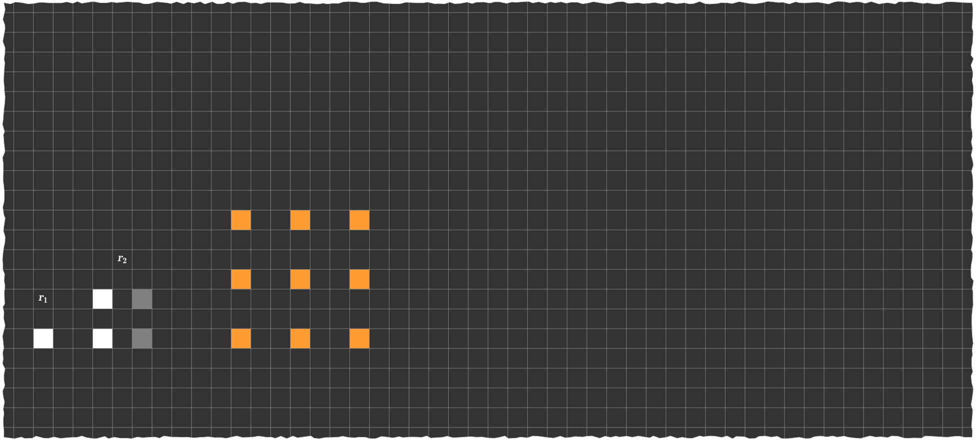

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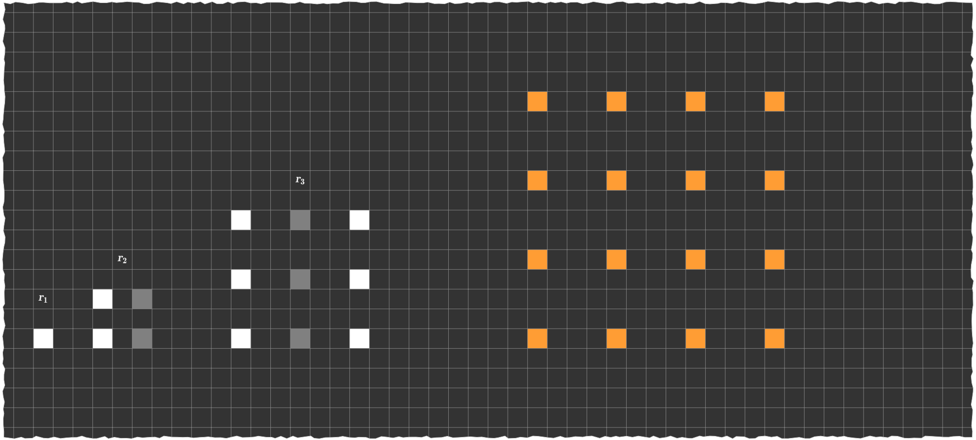

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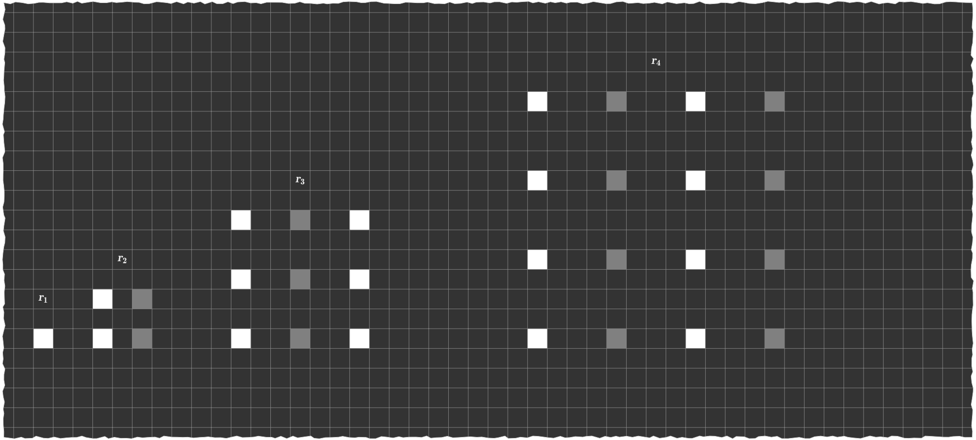

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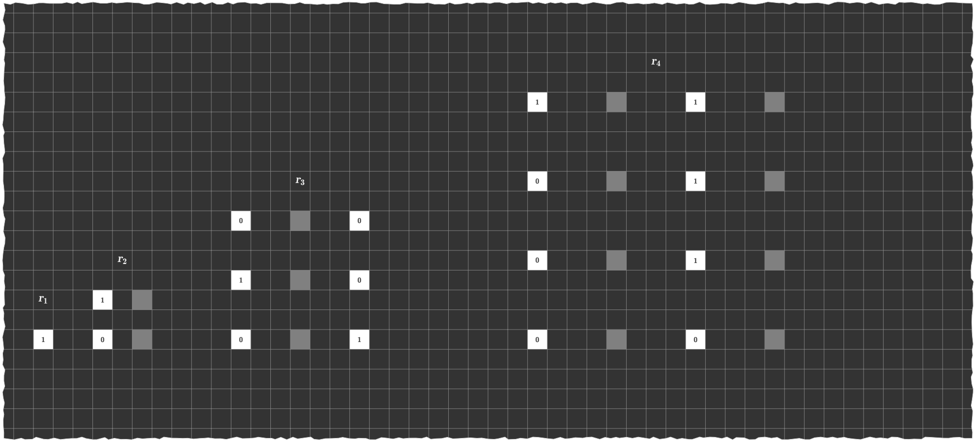

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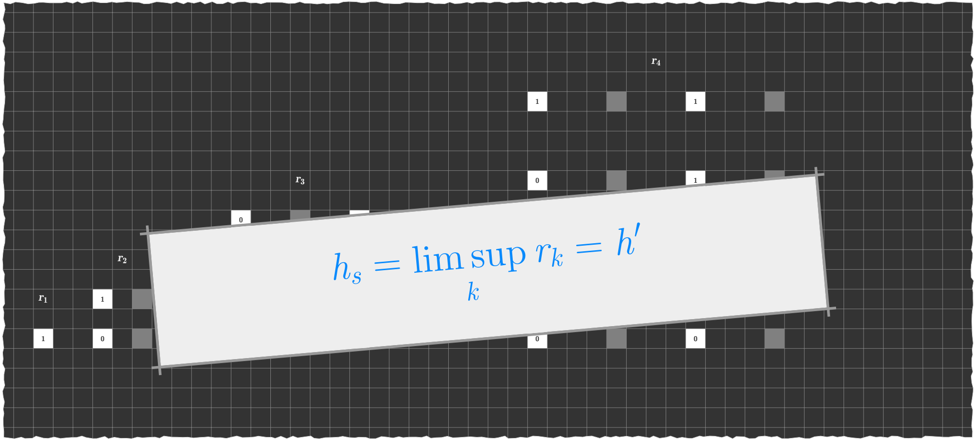
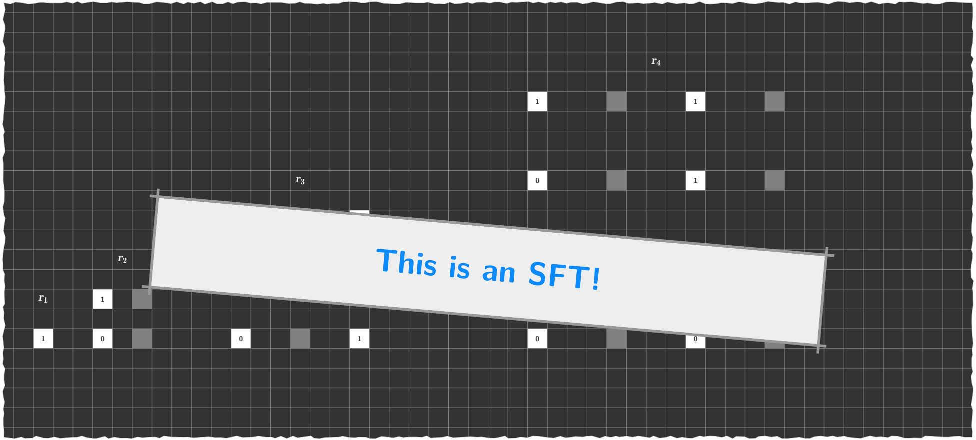


$$h_s = \limsup_k r_k = h'$$

Surface entropies: structure (Part 3)



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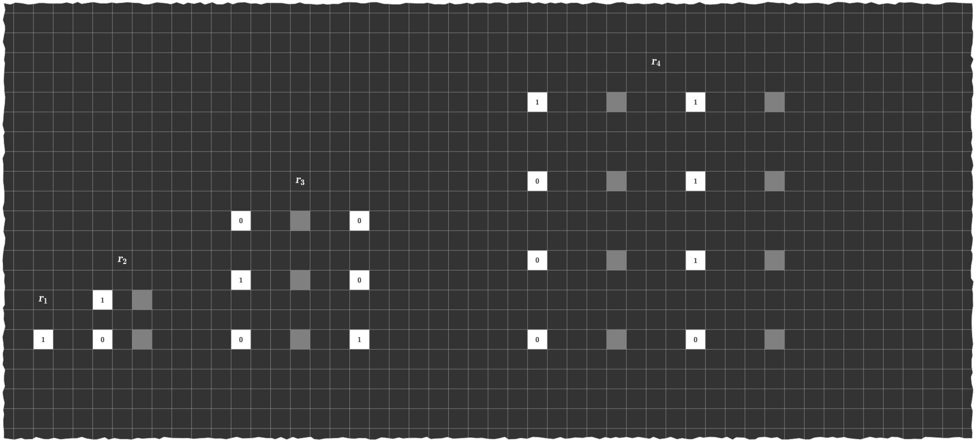

Conclusion

Question:

$$\log N_n \simeq h'n$$

What are the possible values for h' for all SFTs?

Answer:

Surface entropies are exactly the class of Π_3 real numbers!

Thank you

Questions?