

# The aperiodic Domino problem in higher dimension

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STACS 2022, March 15<sup>th</sup>

école —  
normale —  
supérieure —  
paris — saclay —

**LISN** 

# Subshifts

## Definition 1

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A  $\mathbb{Z}^d$  subshift is a set of colorings  $\mathbb{Z}^d \mapsto \Sigma$  defined by forbidden patterns  $\mathcal{F}$ :

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

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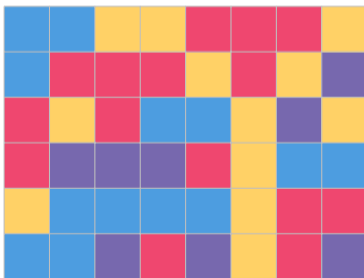
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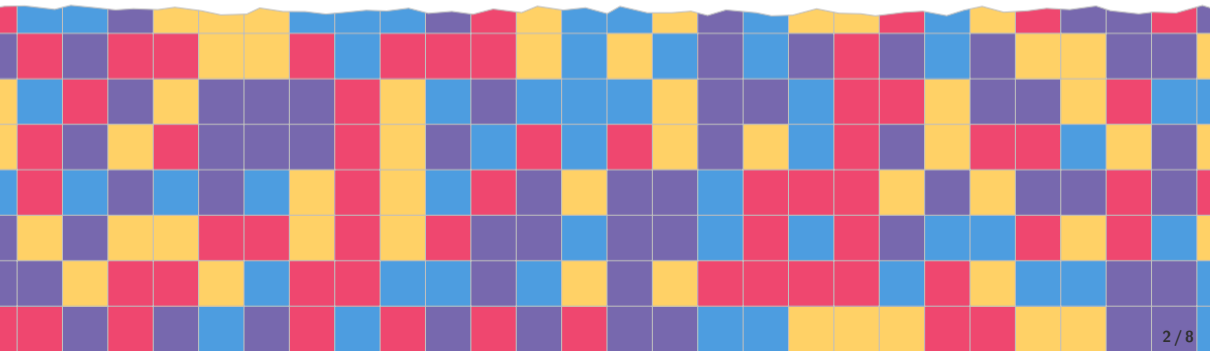
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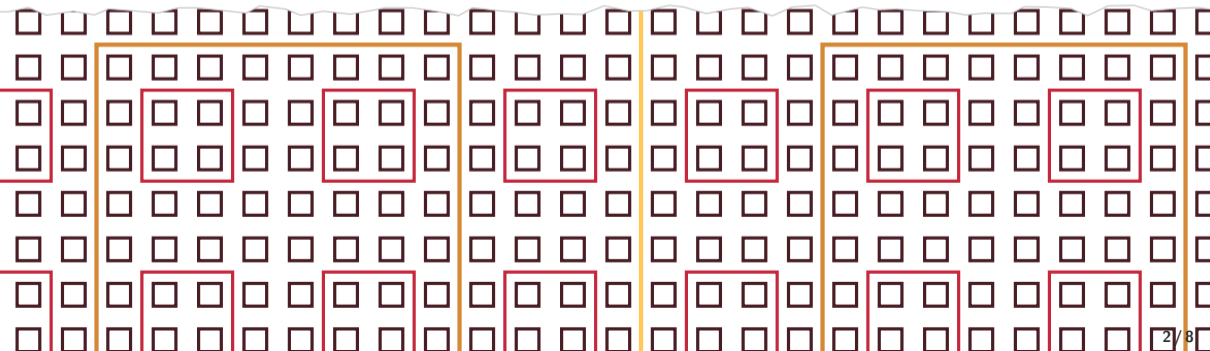
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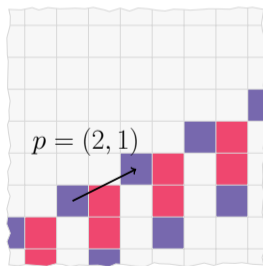
**Aperiodic colorings!**

# Aperiodic Domino problem

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## (A)periodicity

1. A coloring  $x \in \Sigma^{\mathbb{Z}^d}$  is *periodic* of period  $p \in \mathbb{Z}^d$  if:  $\forall i \in \mathbb{Z}^d, x_{i+p} = x_i$ .

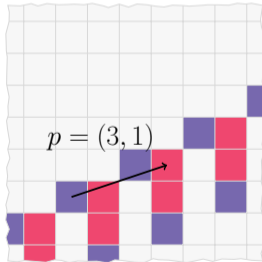


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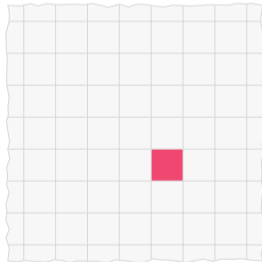


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## Aperiodic Domino problem:

**Input** A SFT (= symbols + finite set of forbidden pattern)

**Output** Is there an admissible *aperiodic* coloring?

Is it undecidable? *How much?*

The background of the slide features a repeating pattern of light blue squares arranged in a grid. Each square contains a light orange line that forms a path, resembling a domino tiling. The paths are interconnected, creating a complex, non-periodic structure that fills the entire page. The text is centered in the middle of the slide.

**Aperiodic Domino problem on  $\mathbb{Z}^2$  [GHV18]**



## [GHV18] On $\mathbb{Z}^2$ : The shepherd of periods

Consider a  $\mathbb{Z}^2$  coloring that is neither  $p_0 = (2, 2)$  nor  $p_1 = (2, 0)$  periodic. Then:



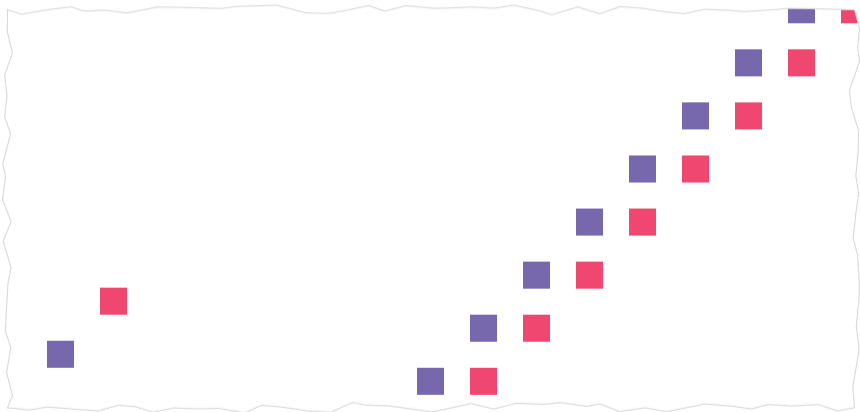
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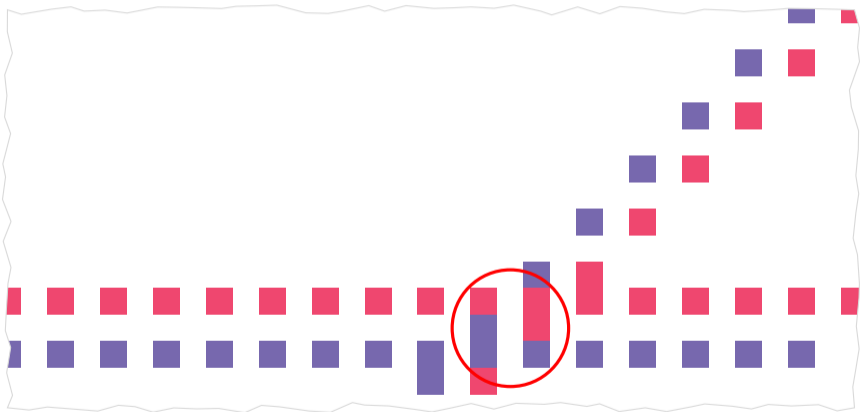
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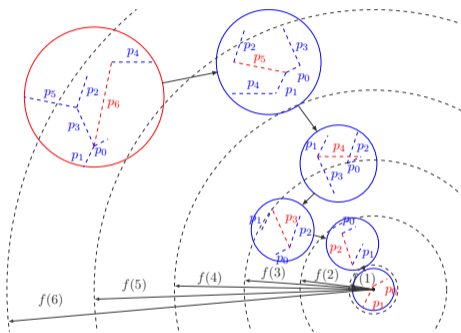
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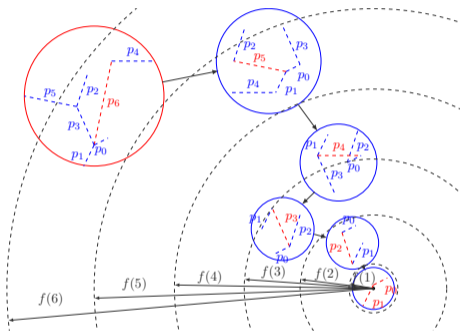
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There exists  $f : \mathbb{N} \mapsto \mathbb{N}$  computable such that, for any  $\mathbb{Z}^2$  coloring:



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**Corrolary 4**

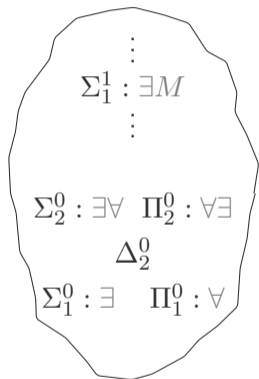
[GHV18, Corollary 8]

Aperiodic Domino is co-recursively enumerable (ie.  $\Pi_1^0$ ) for  $\mathbb{Z}^2$  subshifts.

The background of the slide features a grid of light blue squares. Overlaid on this grid are several orange paths that connect the corners of the squares. These paths form a complex, non-repeating pattern, which is a visual representation of the aperiodic domino problem. The paths are composed of horizontal and vertical segments, with some segments extending across multiple squares. The overall effect is a dense, interconnected network of lines that does not exhibit any simple periodicity.

**Aperiodic Domino problem on  $\mathbb{Z}^d, d \geq 4$**

# Aperiodic Domino in high dimension



**Input** A SFT (= symbols + finite set of forbidden pattern)  
on  $\mathbb{Z}^d$

**Output** Is there an admissible *aperiodic* coloring?

**Theorem 5**

AD in high dimension is not arithmetic

Aperiodic Domino is  $\Sigma_1^1$ -complete for  $\mathbb{Z}^d$  SFTs with  
 $d \geq 4$ .

We reduce **State recurrence**:

**Input** A non-deterministic TM and a state  $q_0$

**Output** Is there a run on  $\varepsilon$  that visits  $q_0$  infinitely often?



# On $\mathbb{Z}^4$ : From AD to state recurrence (Part 1)

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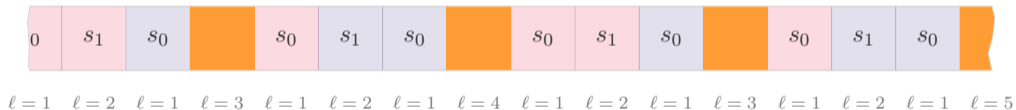
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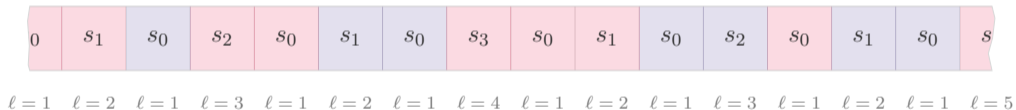
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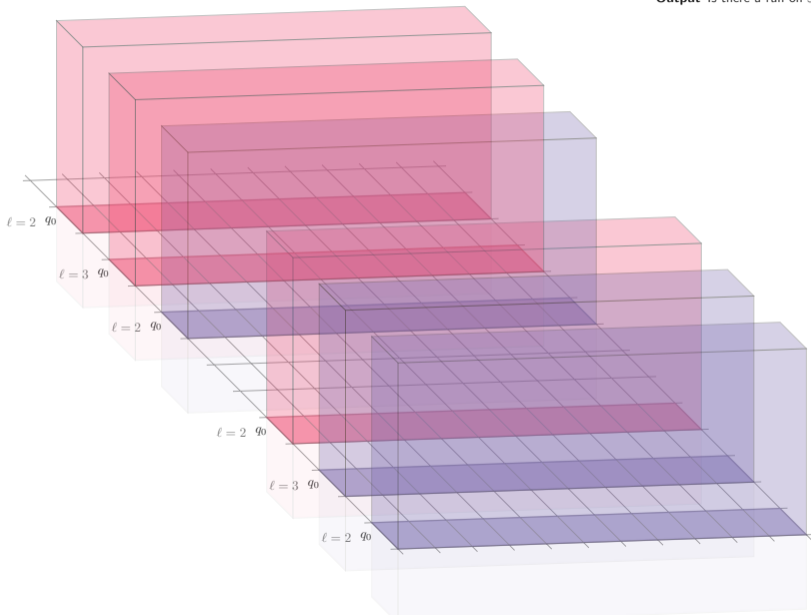
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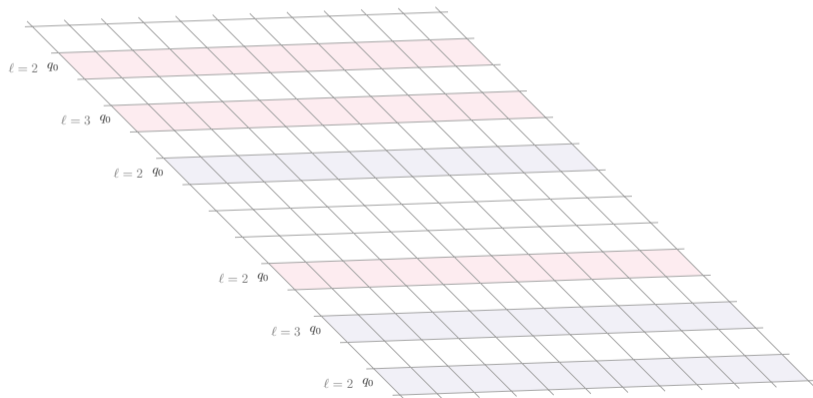


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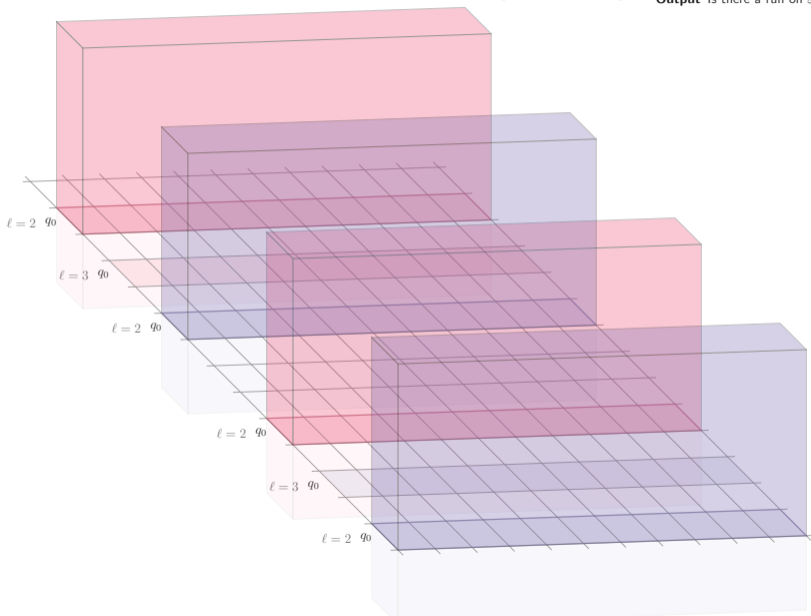


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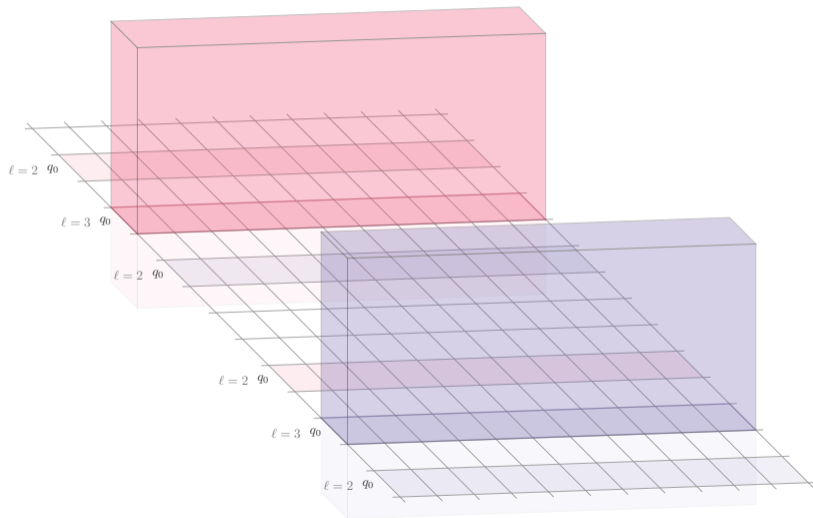


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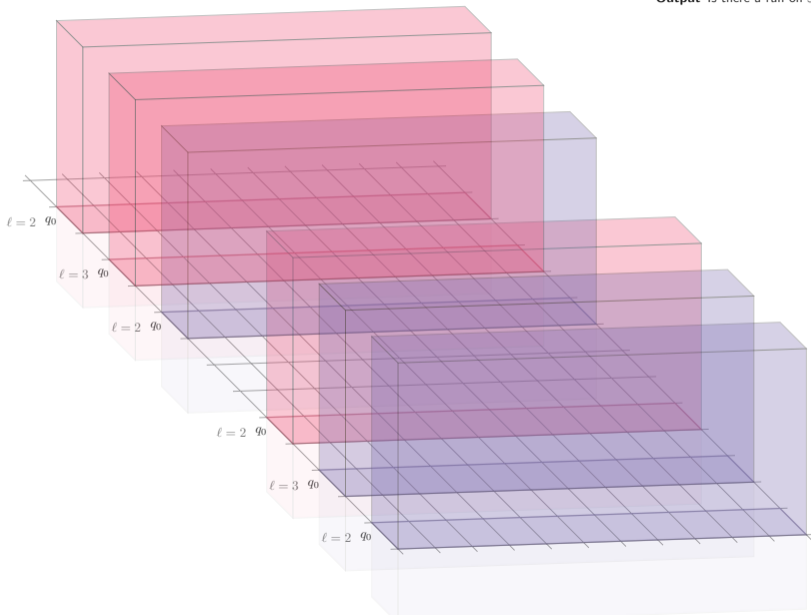


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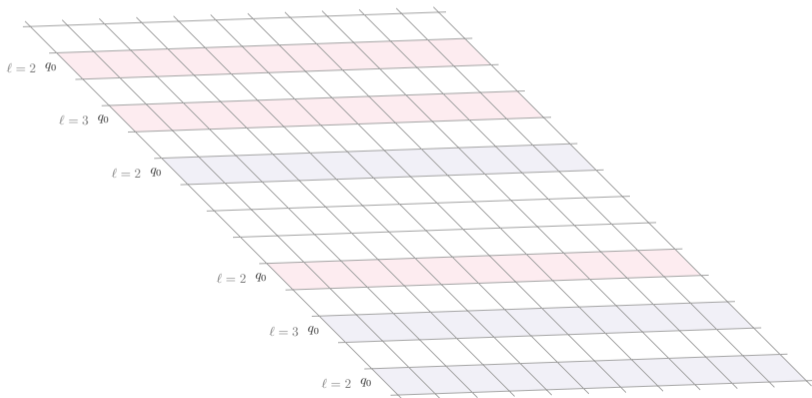


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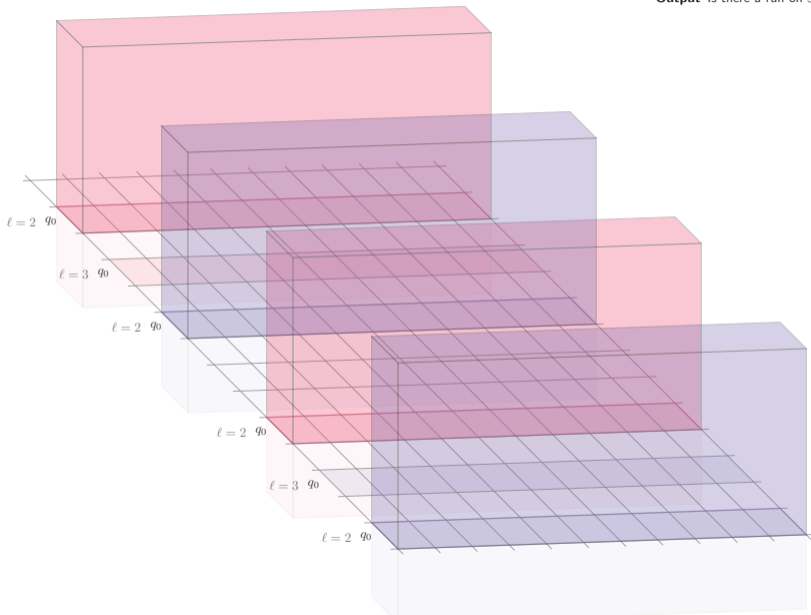


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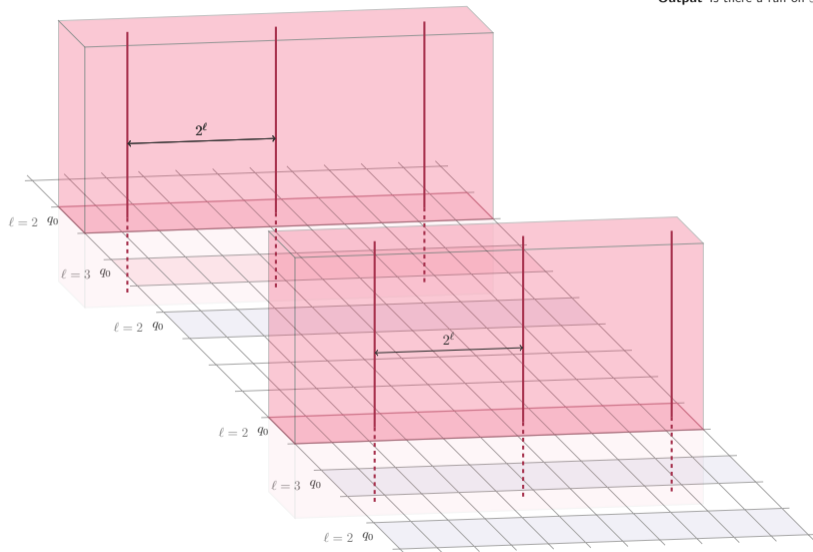


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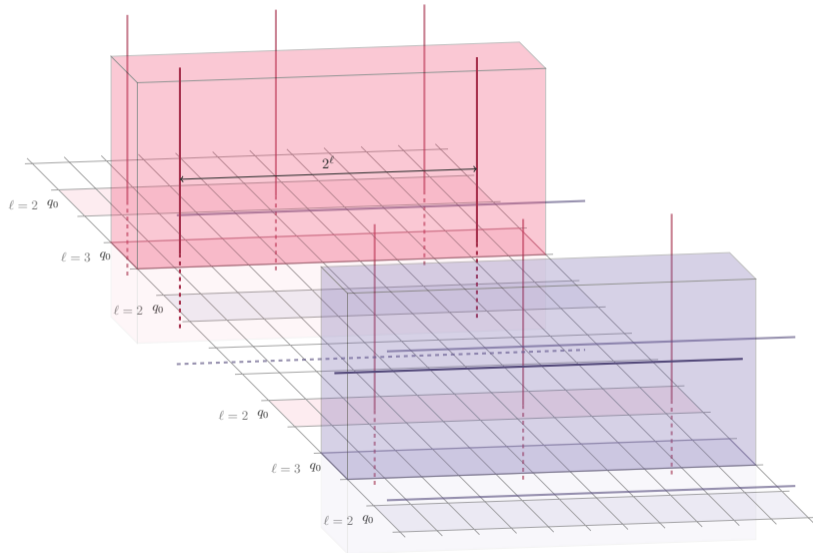




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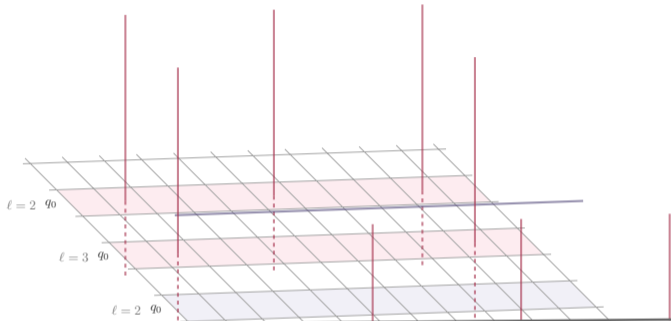


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**Aperiodic coloring  $\iff$  the machine visits  $q_0$  infinitely often.**

Why two additional dimensions? (Spoiler: I lied)

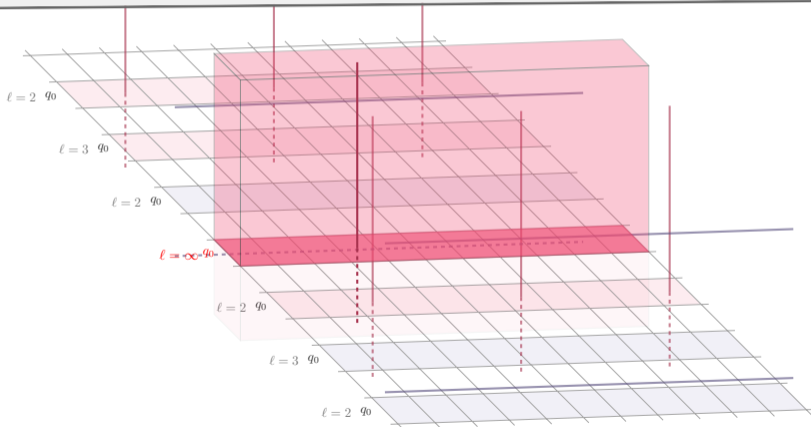
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**Beware the slice of  $\infty$  level!**



# Recap and consequences

## Aperiodic Domino problem:

**Input** A SFT (= symbols + finite set of forbidden pattern)

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Its (computational) complexity depends on the dimension of the subshift.

$\implies$  : separates 2, 3 and 4-dimensional subshifts.

Dimension / type	2D	3D	4D+
finite type	$\Pi_1^0$	<b>open</b>	$\Sigma_1^1$
sofic	$\Pi_1^0$	$\Sigma_1^1$	$\Sigma_1^1$
effective	$\Pi_1^0$	$\Sigma_1^1$	$\Sigma_1^1$

Difficulty of the Domino problem

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Difficulty of the Domino problem

Thank you

Questions?

école \_\_\_\_\_  
normale \_\_\_\_\_  
supérieure \_\_\_\_\_  
paris—saclay \_\_\_\_\_

