The aperiodic Domino problem in higher dimension

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LISN

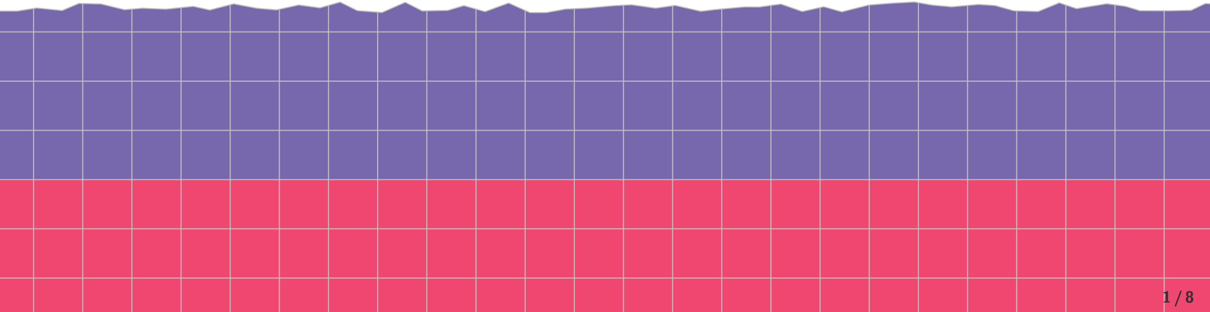
Subshifts

Definition 1 Subshifts

A $\mathbb{Z}^d$ *subshift* is a set of colorings $\mathbb{Z}^d \mapsto \Sigma$ defined by forbidden patterns $\mathcal{F}$:

$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^d} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

$\Sigma = \{$  $\}$ and $\mathcal{F} = \{$  $\}$



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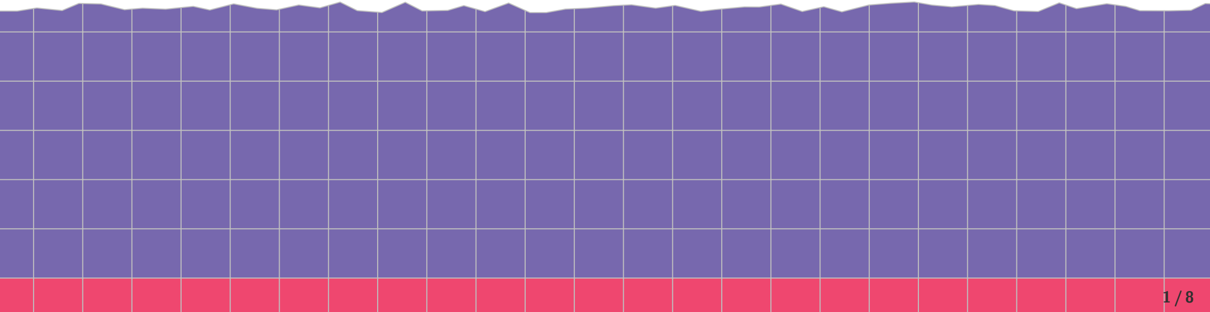

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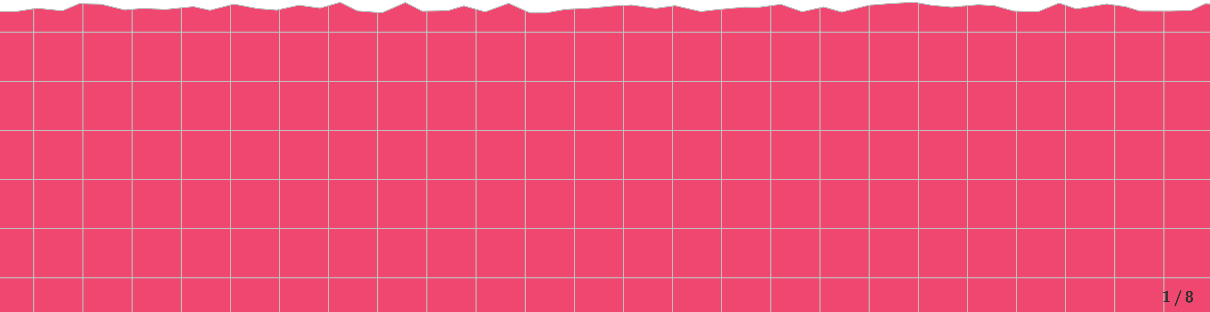

(Classical) Domino problem

Input A SFT (= symbols + finite set of forbidden pattern)
Output Is there an admissible coloring?

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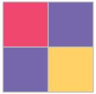
The Domino problem is co-recursively enumerable (ie. Π_1^0).



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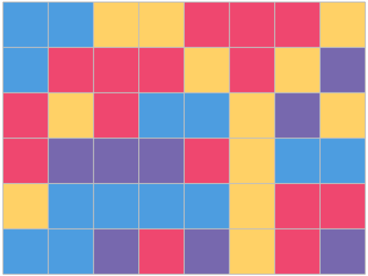

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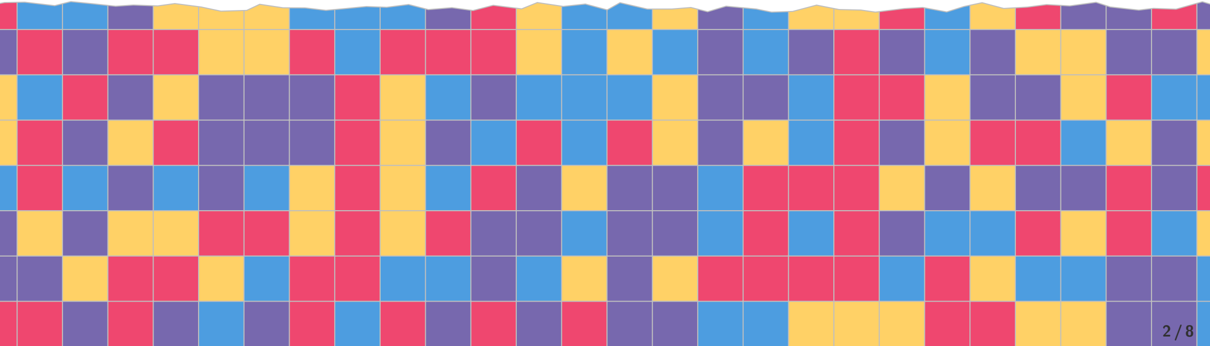

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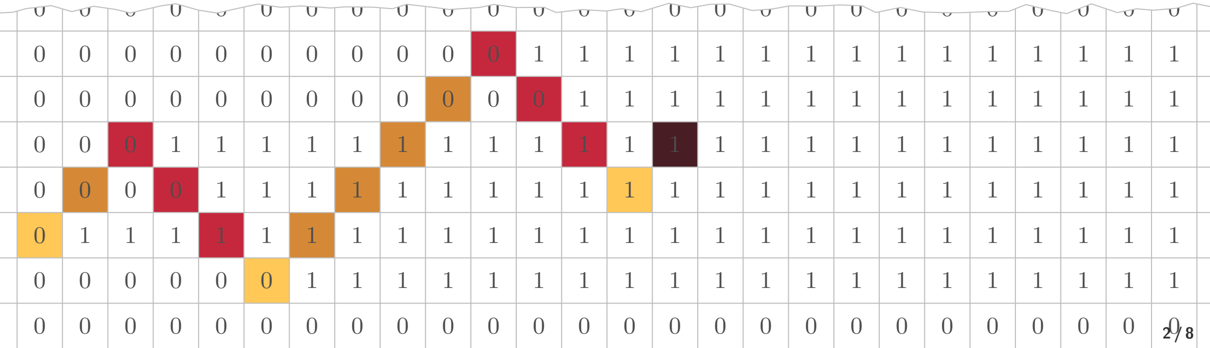

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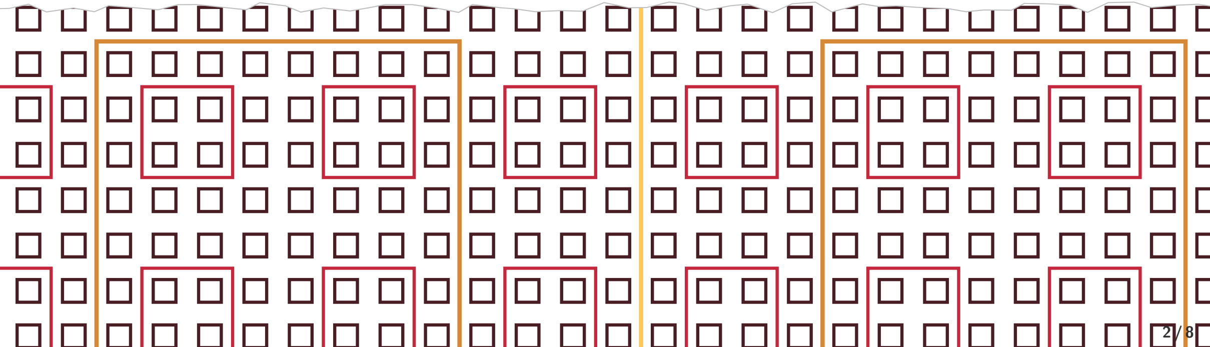

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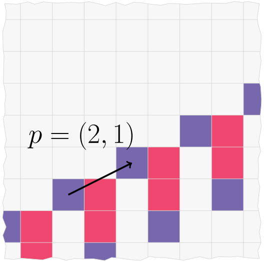
Aperiodic colorings!

Aperiodic Domino problem

Definition 3 (A)periodicity

1. A coloring $x \in \Sigma^{\mathbb{Z}^d}$ is *periodic* of period $p \in \mathbb{Z}^d$ if: $\forall i \in \mathbb{Z}^d, x_{i+p} = x_i$.



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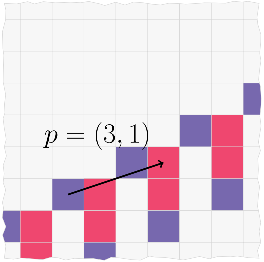

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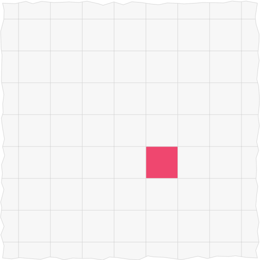

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Aperiodic Domino problem:

Input A SFT (= symbols + finite set of forbidden pattern)

Output Is there an admissible *aperiodic* coloring?

Is it undecidable? *How much?*

Aperiodic Domino problem on $\mathbb{Z}^2$ [GHV18]

[GHV18] On $\mathbb{Z}^2$: The shepherd of periods

Consider a $\mathbb{Z}^2$ coloring that is neither $p_0 = (2, 2)$ nor $p_1 = (2, 0)$ periodic. Then:

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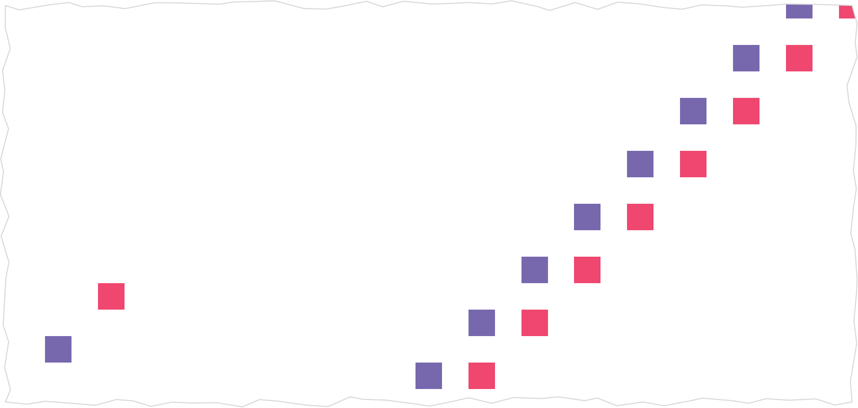

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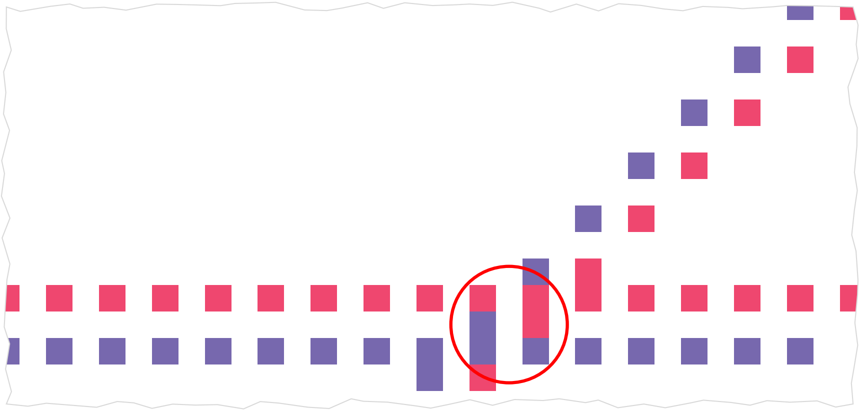

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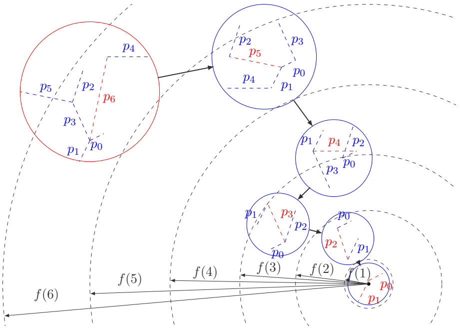
There exists $f : \mathbb{N} \mapsto \mathbb{N}$ computable such that, for any $\mathbb{Z}^2$ coloring:



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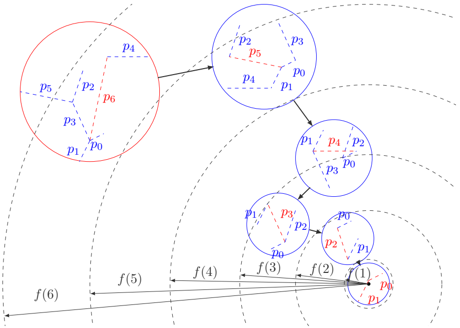
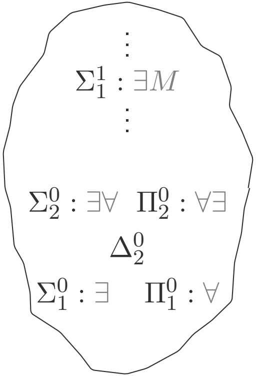


Corrolary 4 [GHV18,Corollary 8]

Aperiodic Domino is co-recursively enumerable (ie. Π^0_1) for $\mathbb{Z}^2$ subshifts.

Aperiodic Domino problem on $\mathbb{Z}^d, d \geq 4$

Aperiodic Domino in high dimension



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Theorem 5 AD in high dimension is not arithmetic

Aperiodic Domino is Σ_1^1-complete for $\mathbb{Z}^d$ SFTs with $d \geq 4$.

We reduce **State recurrence**:

Input A non-deterministic TM and a state q_0

Output Is there a run on ε that visits q_0 infinitely often?

On $\mathbb{Z}^4$: From AD to state recurrence (Part 1)

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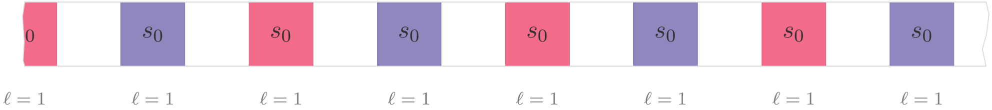

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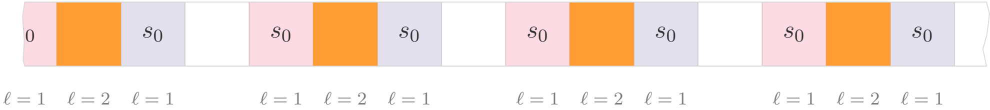

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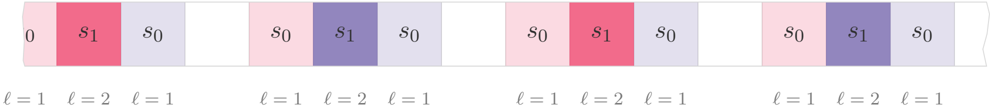

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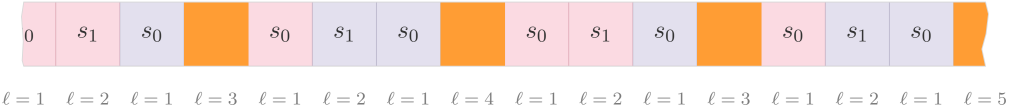

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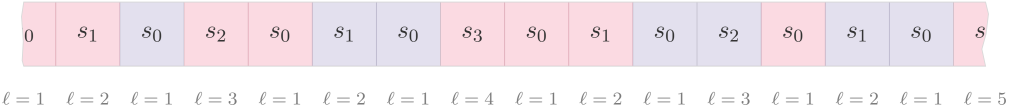

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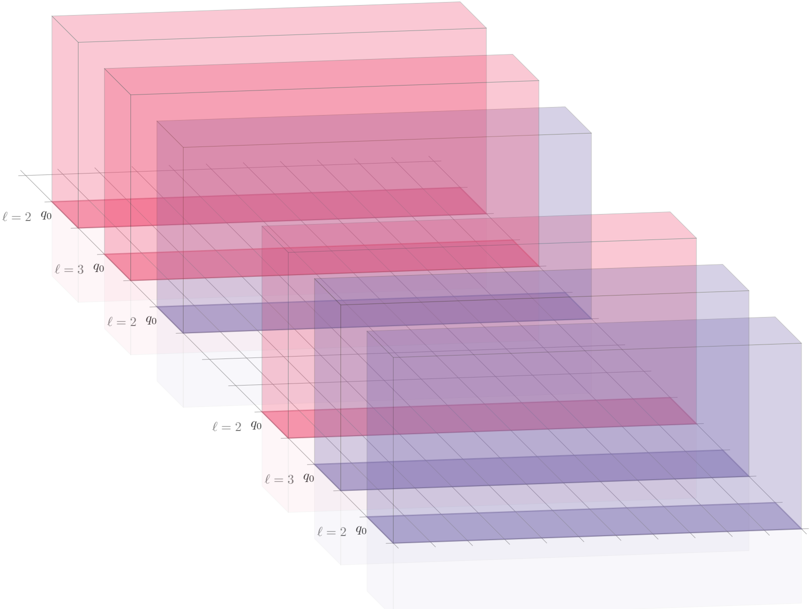

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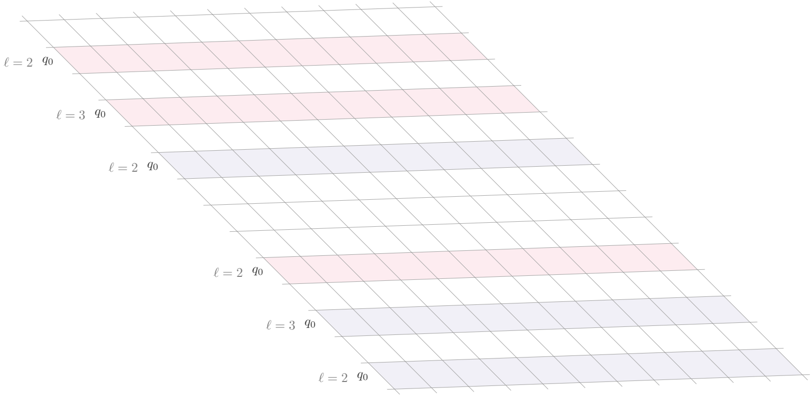

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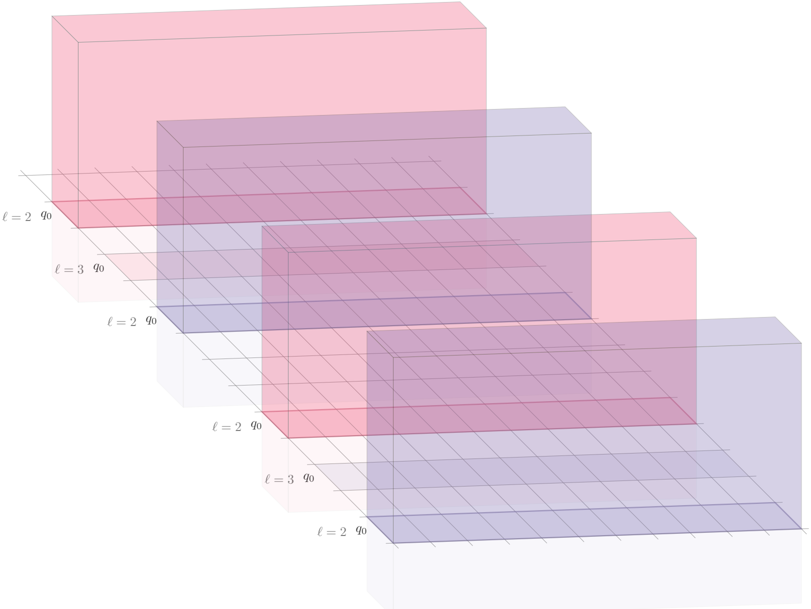

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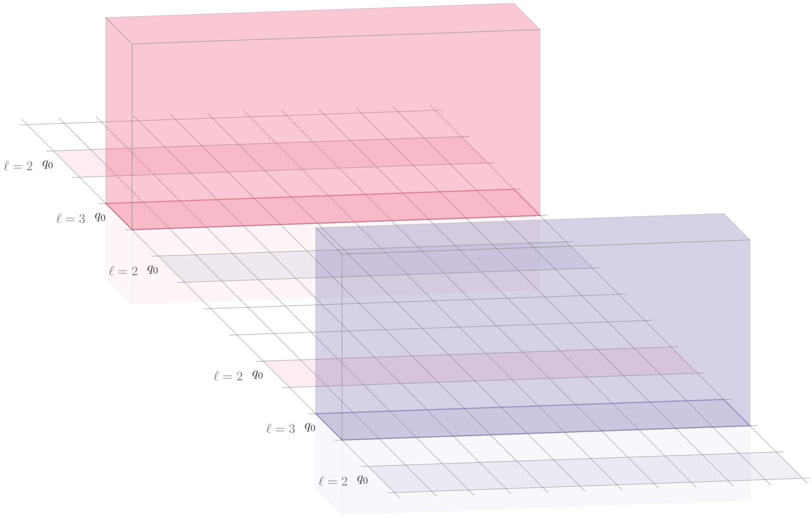

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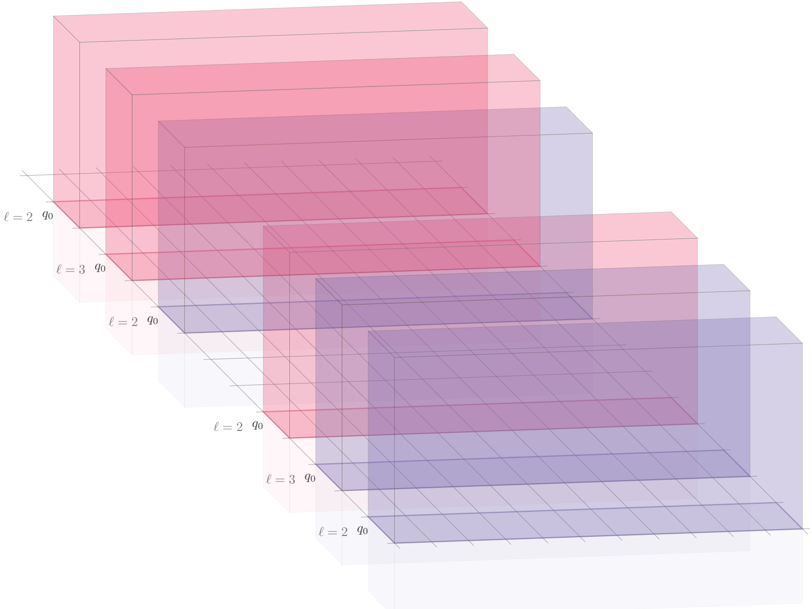

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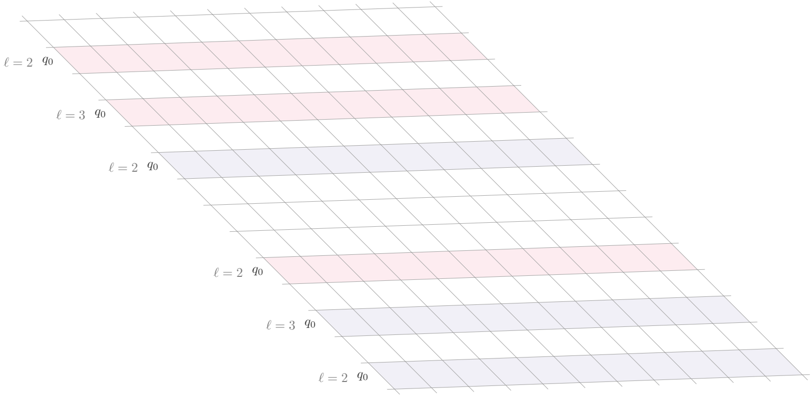

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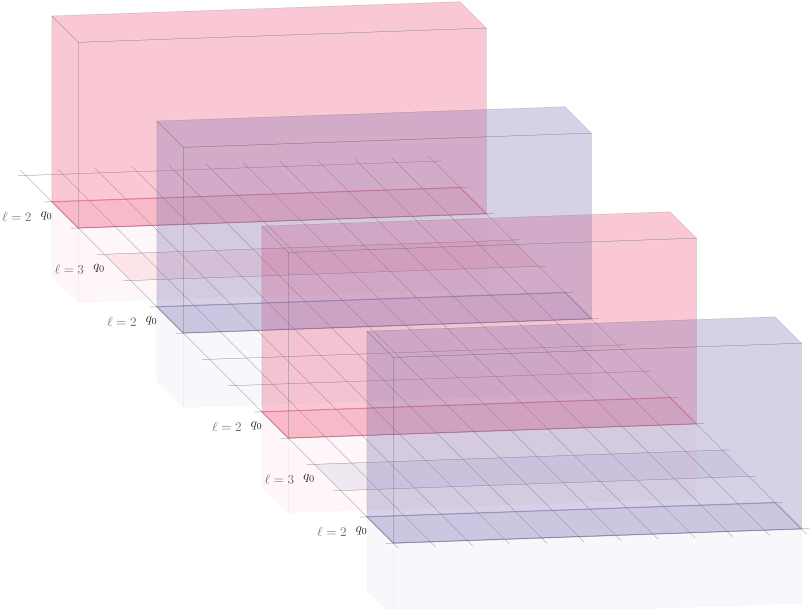

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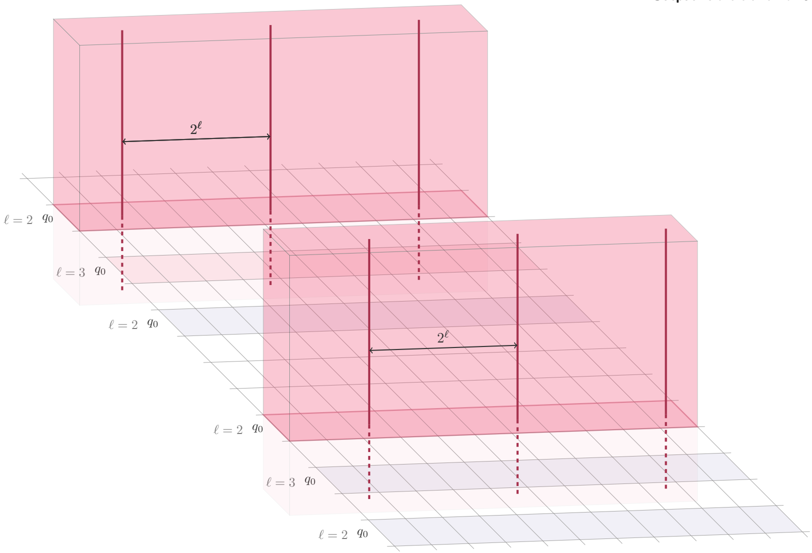

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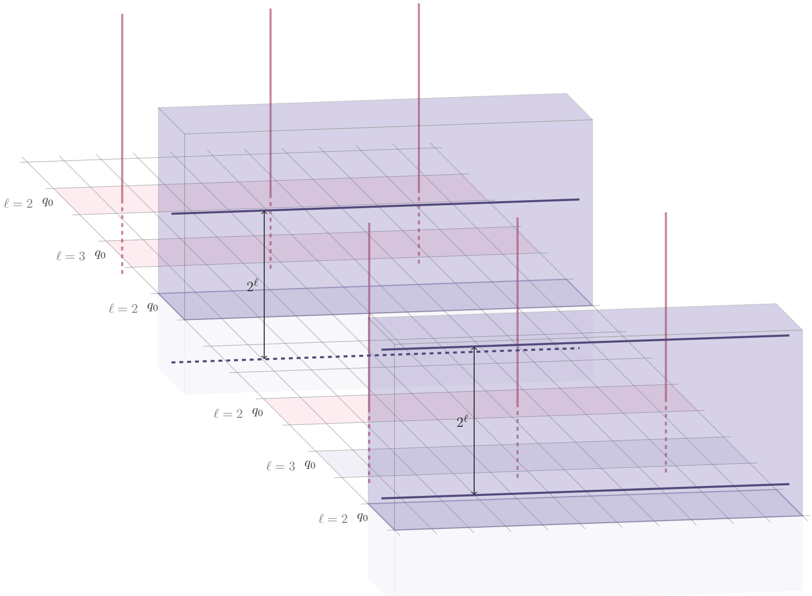

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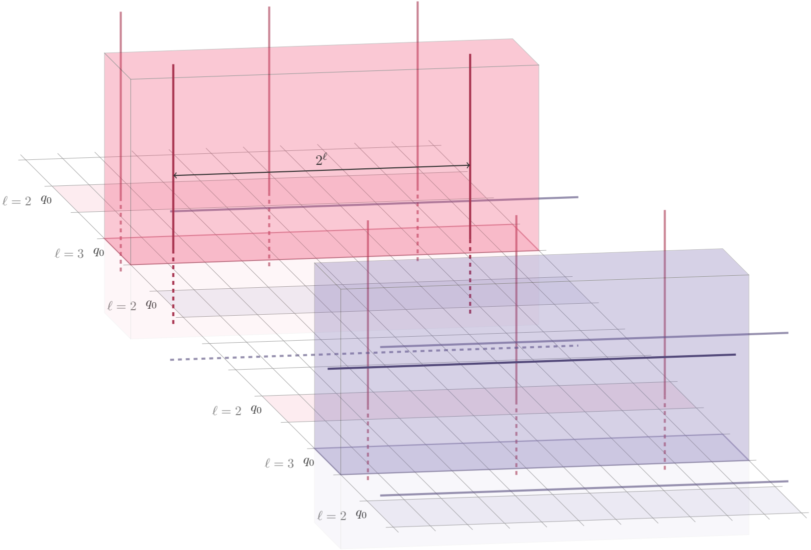

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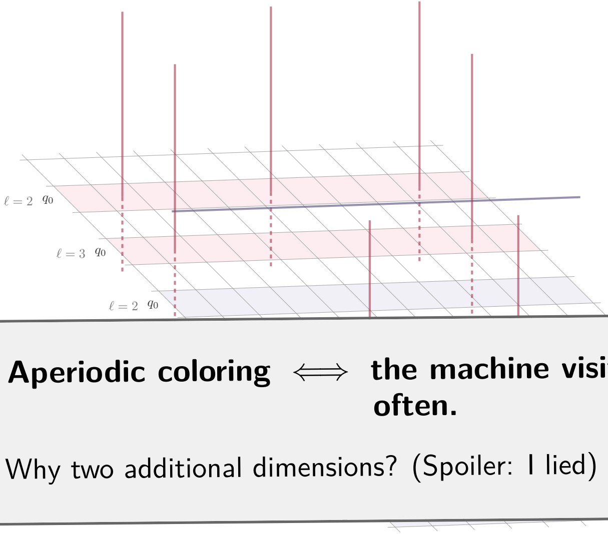

Aperiodic coloring $\Longleftrightarrow$ the machine visits q_0 infinitely often.

Why two additional dimensions? (Spoiler: I lied)

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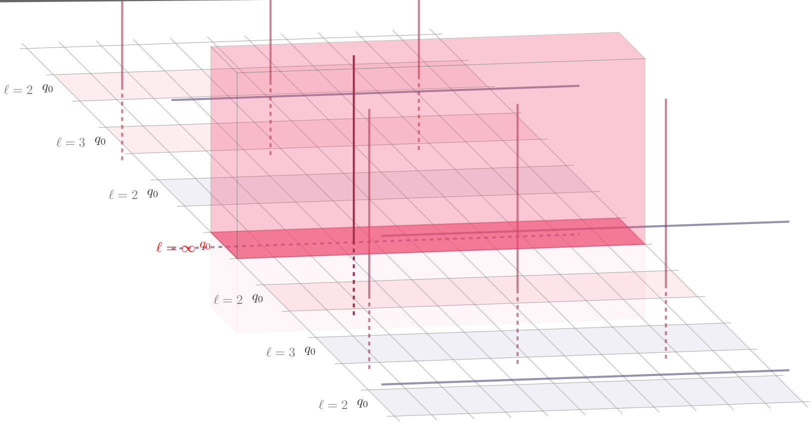
Beware the slice of ∞ level!



Recap and consequences

Aperiodic Domino problem:

Input A SFT ($=$ symbols $+$ finite set of forbidden pattern)

Output Is there an admissible *aperiodic* coloring?

Its (computational) complexity depends on the dimension of the subshift.

$\Longrightarrow$: separates $2, 3$ and 4-dimensional subshifts.

Dimension / type	2D	3D	4D+
finite type	Π_1^0	**open**	Σ_1^1
sofic	Π_1^0	Σ_1^1	Σ_1^1
effective	Π_1^0	Σ_1^1	Σ_1^1

Difficulty of the Domino problem

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Difficulty of the Domino problem

Thank you

Questions?

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normale
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paris—saclay