The aperiodic Domino problem in higher dimension

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Abstract

The classical Domino problem asks whether there exists a tiling in which none of the forbidden patterns given as input appear. In this paper, we consider the aperiodic version of the Domino problem: given as input a family of forbidden patterns, does it allow an aperiodic tiling? The input may correspond to a subshift of finite type, a sofic subshift or an effective subshift.

[7] proved that this problem is co-recursively enumerable(Π_0^1)-complete in dimension 2 for geometrical reasons. We show that it is much harder, namely analytic(Σ_1^1)-complete, in higher dimension: $d \geq 4$ in the finite type case, $d \geq 3$ for sofic and effective subshifts. The reduction uses a subshift embedding universal computation and two additional dimensions to control periodicity.

Such difficulty jumps in decision problems for subshifts usually occur between dimensions 1 and 2; it was unexpected for us to find such a jump is in higher dimension and such a large difficulty gap.

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1 Introduction

Subshifts are sets of colorings (or *configurations*) defined by a family of forbidden patterns. The seminal computational problem on multidimensional subshifts is the *Domino Problem*: given a subshift of finite type, does it contain a configuration? It was proved undecidable on $\mathbb{Z}^2$ in [2, 16] from the construction of *aperiodic subshifts* (subshifts which contain only aperiodic colorings) in which universal computation is embedded. Many similar undecidability results used different subshifts and embeddings to control the structure and properties of their configurations [9, 5, 1, 18, 5] or to characterise the set of possible values of some parameters by computability conditions [11, 13, 4]. These results all rely on the existence of purely aperiodic subshifts on $\mathbb{Z}^d$ for $d \geq 2$ (see [12, Section 1.2] for more details), and show that multidimensional subshifts can be considered as geometrical computational models.

In contrast, topological or geometrical restrictions may lower the "natural" complexity of a problem (compare e.g. [11] with [**?**] or [15]) by breaking our ability to embed computation. In particular, finding the border where the difficulty jump occurs gives a fine understanding of the effect of the restriction [6].

Given the importance of aperiodicity for computation embedding, it is natural to ask the aperiodic version of the Domino problem (**AD**): *given as input a subshift X, is there an aperiodic coloring in X?* It is not difficult to see that this problem is harder than the Domino problem, i.e. co-r.e.(Π_1^0)-hard; however, the natural upper bound is much higher, outside the arithmetical hierarchy.

This question was (to the best of our knowledge) first explored in [7]: the authors proved that **AD** is Π_1^0-complete for $\mathbb{Z}^2$ subshifts[2]. It is an example of problem whose computational complexity is low because of geometrical reasons specific to the two-dimensional case: starting from an aperiodic configuration, we can regroup breaks of periods into concentric balls whose size is controlled by a computable function ([7, Theorem 1]).

In this paper, we study the computational complexity of this problem in higher dimension, where this geometrical property no longer holds (see [7, Section 4] for a counter example). We build an embedding for universal computation that proves that this problem is in a much higher undecidability class – Σ_1^1-complete, its natural upper bound – in sofic subshifts for $d \geq 3$ and in subshifts of finite type for $d \geq 4$.

Our paper is structured as follows.

- In Section 2, we provide definitions for subshifts and the relevant complexity classes;
- In Section 3, we prove that **AD** is Σ_1^1-complete on $\mathbb{Z}^d$ ($d \geq 3$) sofic subshifts;
- In Section 4, we adapt the previous proof to $\mathbb{Z}^d$ ($d \geq 4$) subshifts of finite type;
- In Section 5, we make a side remark relating the existence of an aperiodic configuration in SFTs with their complexity.

We summarize the complexity of **AD** in the following table (new results are highlighted):

Dimension / type	finite type	sofic	effective
2D	Π_1^0-complete	Π_1^0-complete	Π_1^0-complete
3D	open	Σ_1^1-complete	Σ_1^1-complete
4D+	Σ_1^1-complete	Σ_1^1-complete	Σ_1^1-complete

[2] on SFTs, but as [7, Theorem 1] applies to any $\mathbb{Z}^2$ subshift, the result also holds for $\mathbb{Z}^2$ effective subshifts.

There is a border between the space where the complexity of the problem is lowered by geometric properties and the space where computability considerations dominate. For sofic and effective subshifts, this border is the jump between dimensions 2 and 3. For SFTs on $\mathbb{Z}^3$, we conjecture that **AD** is of low undecidability for reasons that are specific to SFTs, and provide a few pointers in conclusion.

2 Definitions and notations

2.1 Subshifts

For a more detailed introduction, we refer the reader to [3, Chapter 9].

Let Σ be a finite alphabet of colors and d a dimension. A *configuration* is a coloring $c : \mathbb{Z}^d \mapsto \Sigma$, and the value of c at position i is denoted c_i. A *pattern* of finite domain $D \subsetneq \mathbb{Z}^d$ is a coloring $w : D \mapsto \Sigma$. We say that a pattern w appears in a configuration x and write $w \sqsubseteq x$ if $w_j = x_{i+j}$ for some $i \in \mathbb{Z}^d$ and all $j \in D$. Given a configuration x and a vector $t \in \mathbb{Z}^d$, denote $\sigma^t(x)$ the shift of x by t: for any $i \in \mathbb{Z}^d, \sigma^t(x)_i = x_{i-t}$.

▶ **Definition 1** (Periodicity). **1.** *In a configuration $x \in \Sigma^{\mathbb{Z}^d}$, a vector $p \in \mathbb{Z}^d$ is* broken at position i *if $x_{i+p} \neq x_i$.*

2. *A configuration $x \in \Sigma^{\mathbb{Z}^d}$ is (strongly)* aperiodic *if every vector $p \in \mathbb{Z}^d$ is broken in x.*

In the following definition, Σ is equipped with the discrete topology and $\Sigma^{\mathbb{Z}^d}$ with the product topology. $\Sigma^{\mathbb{Z}^d}$ is then a Cantor space.

▶ **Definition 2** (Subshifts). *A* subshift *is a closed and σ-invariant subset of $\mathbb{Z}^d$. Equivalently, there is a family of forbidden patterns $\mathcal{F}$ such that*

$$X = X_{\mathcal{F}} := \left\{x \in \Sigma^{\mathbb{Z}^d} : \forall w \in \mathcal{F}, w \not\sqsubseteq x\right\}$$

Two families of forbidden patterns may define the same subshift.

▶ **Definition 3** (Classes of subshifts). *A subshift $Y \subseteq \Sigma^{\mathbb{Z}^d}$ is:*

1. of finite type *(*SFT*) if it can be defined by a finite family of forbidden patterns.*

2. sofic *if there exists an SFT $X \subseteq \Sigma'^{\mathbb{Z}^d}$ and a projection $\pi : \Sigma' \mapsto \Sigma$ such that $Y = \pi(X)$.*

3. effective *if it can be defined by a recursively enumerable family of forbidden patterns.*

SFTs are of course sofic and sofic subshifts are effective. On the other direction, a result from [9] (improved in [5, 1]) proves that effective subshifts are sofic in higher dimension. More precisely, $X^{\uparrow}$ is a $(d+k)$-dimensional *lift* of a subshift $X \subseteq \Sigma^{\mathbb{Z}^d}$ if its configurations are configurations of X repeated along the k additional dimensions. Then:

▶ **Theorem 4** ([5, 1]). *For any $\mathbb{Z}^d$ effective subshift X, its $(d+1)$-dimensional lifts are sofic.*

2.2 Hierarchy of undecidability

Many-one reductions define a preorder on decision problems ("P_1 is easier than P_2"), so we can define hierarchies according to "how far" a problem is from being computable.

Arithmetical hierarchy

Starting from recursively enumerable (Σ_1^0) and co-recursively enumerable (Π_1^0) problems, the arithmetical hierarchy progressively defines higher levels of undecidability.

▶ **Definition 5** (Arithmetical hierarchy). *For a decision problem $P : \mathbb{N} \mapsto \{0,1\}$ and $m \geq 1$,*

1. *$P \in \Sigma_m^0$ if there is a computable relation $R(n, k_1, ..., k_m)$ such that*

$$P(n) = 1 \Leftrightarrow \exists k_1, \forall k_2, \exists k_3, \dots R(n, k_1, \dots, k_m).$$

2. *$P \in \Pi_m^0$ if this definition holds when swapping $\forall$ and $\exists$ quantifiers.*

P is arithmetical *if it belongs to a level of this hierarchy.*

As $\Sigma_m^0 \cup \Pi_m^0 \subseteq \Sigma_{m+1}^0 \cap \Pi_{m+1}^0$, this indeed defines a hierarchy. For more details, we refer the reader to [17, Chapter 4].

Analytical hierarchy

Above the arithmetical hierarchy, the analytical hierarchy allows for second-order quantifications on sets. Here we need only the first level.

▶ **Definition 6** (Class Σ_1^1). *A decsion problem $P : \mathbb{N} \mapsto \{0,1\}$ is Σ_1^1 if there exists an* arithmetical *relation R such that*

$$P(n) = 1 \Leftrightarrow \exists f \in 2^{\mathbb{N}}, R^f(n)$$

in which R^f denotes the relation R with f given as an oracle.

All arithmetical sets are Σ_1^1. In terms of computational power, Σ_1^1 sets are (a lot) harder than arithmetical sets: to make an analogy between computability and topology, if Σ_1^0 sets correspond to the open sets, then Σ_1^1 sets are not even Borel. For more details, see [14, Chapter IV.2]. A typical example of a Σ_1^1-*complete problem* is the following:

▶ **Theorem 7** (State Recurrence [8, Corollary 6.2]). *The problem of* ***State Recurrence (SR)****:*
Input: *A nondeterministic Turing machine (NTM) $\mathcal{M}$, and one of its states q_0.*
Output: *Is there a run of $\mathcal{M}$ on the empty input ε in which q_0 is visited infinitely often?*
is a Σ_1^1-complete problem.

2.3 The aperiodic Domino (AD) problem and its complexity

▶ **Definition 8** (Aperiodic Domino problem (**AD**)).
Input: *An effective family of d-dimensional patterns.*
Output: *Is there an aperiodic configuration in the effective subshift $X_{\mathcal{F}}$?*
We consider variations of **AD** depending on the type of input subshift (SFT, sofic, effective). There are natural lower and upper bounds on the complexity of **AD** that do not depend on the input type:

▶ **Proposition 9.** ***AD*** *is Π_1^0-hard for $\mathbb{Z}^d$ subshifts ($d \geq 2$).*

Proof. We reduce the Domino problem to **AD**. Let Y be a $\mathbb{Z}^d$-SFT with only aperiodic configurations (see e.g. [16]). For any $\mathbb{Z}^d$ subshift X, the cartesian product $X \times Y$ has the same type (SFT, sofic, effective), has only aperiodic configurations, and is non-empty if and only if X is non-empty. ◀

▶ **Proposition 10.** ***AD*** *is a* Σ_1^1 *problem for* $\mathbb{Z}^d$ *subshifts.*

Proof. Let $\mathcal{F}$ be the effective family of forbidden patterns given as input. The existence of an aperiodic configuration can be written as:

$$\exists x \in \Sigma^{\mathbb{Z}^d}, x \in X_{\mathcal{F}} \text{ and } x \text{ is aperiodic.}$$

Taking any computable encoding between $\Sigma^{\mathbb{Z}^d}$ and $2^{\mathbb{N}}$, we can see that the first (existential) quantifier is of second order and can be written as a quantifier on $2^{\mathbb{N}}$. The rest of the expression is a Π_2^0 relation, and in particular arithmetical (x being given as oracle):

- $x \in X_{\mathcal{F}} \Leftrightarrow \forall w \in \mathcal{F}, \forall i \in \mathbb{Z}^d, x_{|i+\mathrm{dom}(w)} \neq w$;
- x is aperiodic $\Leftrightarrow \forall p \in \mathbb{Z}^d, \exists i \in \mathbb{Z}^d, x_i \neq x_{i+p}$. ◀

3 Σ_1^1-completeness for $\mathbb{Z}^3$ sofic and effective subshifts

▶ **Theorem 11.** ***AD*** *for* $\mathbb{Z}^3$ *sofic subshifts is a* Σ_1^1*-complete problem.*

To prove this theorem, we reduce (many-one reduction) the problem **SR**. Let $\mathcal{M}$ be some nondeterministic Turing machine (NTM) and q_0 one of its states. We create a sofic subshift Y_3 which contains an aperiodic configuration if and only if $\mathcal{M}$ admits a run from the empty word which visits q_0 infinitely often. The proof is divided in three parts:

1. Section 3.1: creation of an auxiliary $\mathbb{Z}$ Toeplitz subshift $T_{\mathcal{M}}$;
2. Section 3.2: creation of Y_3 and proof that it is sofic;
3. Section 3.3: proof that $Y_3 \in$ **AD** iff $(\mathcal{M}, q_0) \in$ **SR**.

3.1 $T_{\mathcal{M}}$: $\mathbb{Z}$ Toeplitz corresponding to state sequences of $\mathcal{M}$

In this section, we transform the set of sequences of states in all the runs of $\mathcal{M}$ into a $\mathbb{Z}$ subshift with a convenient structure called *Toeplitz*.

$\mathbb{Z}$ Binary Toeplitz subshift X_T

Consider the substitution σ on the alphabet $\{{\color{gray}\rightarrow}, {\color{gray}\leftarrow}, \rightarrow, \leftarrow\}$:

$$\sigma({\color{gray}\rightarrow}) = \rightarrow{\color{gray}\leftarrow} \qquad \sigma({\color{gray}\leftarrow}) = \leftarrow{\color{gray}\leftarrow} \qquad \sigma(\rightarrow) = \rightarrow{\color{gray}\rightarrow} \qquad \sigma(\leftarrow) = \leftarrow{\color{gray}\rightarrow}$$

Define the $\mathbb{Z}$-subshift X_σ by forbidding every pattern that does not appear in $\sigma^\omega(\rightarrow)$ (any other seed symbol would yield the same subshift).

$$\sigma^\omega(\rightarrow) = \rightarrow{\color{gray}\rightarrow}\rightarrow{\color{gray}\leftarrow}\rightarrow{\color{gray}\rightarrow}\leftarrow{\color{gray}\leftarrow}\rightarrow{\color{gray}\rightarrow}\rightarrow{\color{gray}\leftarrow}\leftarrow{\color{gray}\rightarrow}\leftarrow{\color{gray}\leftarrow} \ \ldots$$

▶ **Definition 12** (Binary Toeplitz subshift)**.** *The* binary Toeplitz subshift X_T *is the color-erasing projection of* X_σ *on the alphabet* $\{\rightarrow, \leftarrow\}$.

This subshift has the following structure: in a configuration of X_T,

Level 1 One position out of two has an alternating sequence of $\rightarrow$ and $\leftarrow$;

Level 2 One position out of two *in the remaining positions* (i.e. one out of four) has an alternating sequence of $\rightarrow$ and $\leftarrow$, etc.

More generally, a position i is of level n if it is 2^{n+1} periodic (cells at positions $i + k2^{n+1}$ all have the same value). In a configuration of X_T, there may exist at most one position i which does not have a level (i.e. it is not 2^{n+1} periodic for any n): we say that $\text{level}_z(i) = \infty$. In a finite pattern of size between 2^n and 2^{n+1}, one can determine all levels $\leq n-1$.

$T_{\mathcal{M}}$: $\mathbb{Z}$-Toeplitzification of sequences of states of $\mathcal{M}$

▶ **Definition 13** (Toeplitzification of a set of sequences). *Given a set of sequences $A \subseteq \Sigma^{\mathbb{N}}$, we define the corresponding Toeplitzified subshift T_A on the alphabet $\Sigma \times \{\rightarrow, \leftarrow\}$ as:*

$$T_A = \left\{ (x,z) \in (\Sigma \times \{\rightarrow, \leftarrow\})^{\mathbb{Z}} : \begin{array}{l} z \in X_T, \exists (a_n)_{n\in\mathbb{N}} \in A, \\ \forall i \in \mathbb{Z}, \mathrm{level}_z(i) = n \implies x_i = a_n \end{array} \right\}$$

Note that a position of infinite level may be marked with any symbol of Σ. We cannot force this symbol without breaking the next lemma.

Now take $\Sigma = Q$, the set of states of $\mathcal{M}$, and define $S_{\mathcal{M}}$ as the set of sequences $(s_t)_{t\in\mathbb{N}}$ on the alphabet Σ such that there exists a non-terminating run of $\mathcal{M}$ from the empty input whose state at time t is s_t. Let $T_{\mathcal{M}} := T_{S_{\mathcal{M}}}$ be its Toeplitzification.

▶ **Lemma 14.** *$T_{\mathcal{M}}$ is a $\mathbb{Z}$ effective subshift.*

Proof. This stems from the fact that the set of prefixes of $S_{\mathcal{M}}$ is computable: for any $n \geq 0$, we can enumerate all oracles of non-determinism of $\mathcal{M}$ of size $\leq n$ and compute S_n, the set of finite prefixes of length n in $S_{\mathcal{M}}$.

Consider the following algorithm that defines a family of forbidden patterns. For all n:

- Compute the globally admissible patterns of X_T of size $2^n + 1$; (*Note that the language of patterns of X_T is computable: it is both recursively and co-recursively enumerable.)*
- Compute $\mathcal{S}_n$;
- Forbid all patterns $(u, v) \in (\Sigma \times \{\rightarrow, \leftarrow\})^{2^n+1}$, except if v is a pattern in X_T and there exists a prefix $(s_t)_{0\leq t\leq n} \in \mathcal{S}_n$ such that:

$$\forall i, j \in \mathbb{Z}, \ (\mathrm{level}_v(i) = \mathrm{level}_v(j) \leq n) \implies u_i = u_j = s_{\mathrm{level}_v(i)}.$$

This procedure defines an effective subshift E. We prove $E = T_{\mathcal{M}}$. Indeed:

$E \subseteq T_{\mathcal{M}}$ Take $(u, v) \in E$ and $(u^n, v^n) = (x, z)[-2^n, 2^n]$. By definition of E, there exists a finite prefix $s^n \in S_n$ such that for any positions i, j with $\mathrm{level}_{v^n}(i) = \mathrm{level}_{v^n}(j) = \ell$, we have $u^n_i = u^n_j = s_\ell$. This sequence of prefixes is increasing, so it converges towards some sequence $s \in S_{\mathcal{M}}$. Then for any $i, j \in \mathbb{Z}$ such that $\mathrm{level}_v(i) = \mathrm{level}_v(j) = \ell < +\infty$, we have $x_i = x_j = s_\ell$. So $(x, z) \in T_{\mathcal{M}}$.

$T_{\mathcal{M}} \subseteq E$ No pattern forbidden in the algorithm appears in any configuration of $T_{\mathcal{M}}$. ◀

3.2 Y_3: the desired $\mathbb{Z}^3$ subshift

We create a subshift Y_3 which contains an aperiodic configuration if and only if there exists a run of $\mathcal{M}$ on the empty word which visits q_0 infinitely often.

Some intuition on Y_3

As one might expect, each configuration of Y_3 contains the lift of a configuration of $T_{\mathcal{M}}$ corresponding to a run of $\mathcal{M}$ on the empty word. We add lines to make it aperiodic if and only if q_0 appears infinitely often. However, compactness leads to issues:

- Every decision of breaking periods must occur locally, without the ability to know whether the number of visits of q_0 is finite or infinite. Indeed, visits of q_0 can occur arbitrarily late, so compactness could lead to a position of finite level “believing” that q_0 is visited infinitely often in the future, even though it is not.
- Since breaks of periods are decided locally, we must break increasingly large periods according to the level of a position in the Toeplitz sequence. However, by compactness, the position of infinite level is able to break periods of every size by itself.

This explains why we need to add two dimensions to $T_{\mathcal{M}}$: each position in $T_{\mathcal{M}}$ is periodic in one dimension, and break periods in the other. This way, the single position of infinite level may break horizontal or vertical periods, but not both.

Effective 2D subshifts: $Y_2^{\rightarrow}$ and $Y_2^{\leftarrow}$

A configuration of $Y_2^{\rightarrow}$ is composed of three layers:

- Layer 1 & 2 : it contains a $\mathbb{Z}^2$ lift of a configuration $x' \in T_{\mathcal{M}}$. That is, $\forall i, j : x_{i,j} = x'_i$.
- Layer 3: on the alphabet $\{\blacksquare, \square\}$. For every ℓ and in every column of level ℓ containing $(q_0, \rightarrow)$ on Layers 1 and 2, Layer 3 contains regularly placed $\blacksquare$ cells separated by 2^ℓ $\square$ cells. Every other cell contains $\square$ on Layer 3.

Formally, $Y_2^{\rightarrow}$ can be written as:

$$\left\{\begin{array}{l} x \in (\Sigma \times \{\rightarrow, \leftarrow\} \times \{\blacksquare, \square\})^{\mathbb{Z}^2} : \pi_{1,2}(x) \in L_{1,2}, \\ \qquad\qquad\qquad\qquad\qquad\qquad \exists z \in X_T, \pi_2(x) \text{ is a } \mathbb{Z}^2 \text{ lift of } z, \\ \forall i, j \in \mathbb{Z}, \\ \qquad x_{i,j} = (\cdot, \cdot, \blacksquare) \implies \forall j', x_{i,j'} = (q_0, \rightarrow, \cdot) \\ \qquad x_{i,j} = (\cdot, \cdot, \blacksquare) \text{ and } \mathrm{level}_z(i) = \ell \implies \left(\forall j', x_{i,j'} = (\cdot, \cdot, \blacksquare) \iff j' = j + m2^\ell\right) \end{array}\right\}$$

Its counterpart $Y_2^{\leftarrow}$ is defined similarly by replacing $\rightarrow$ by $\leftarrow$ in the previous definition. It is clear that both $Y_2^{\rightarrow}$ and $Y_2^{\leftarrow}$ are effective $\mathbb{Z}^2$ subshifts.

Sofic 3D subshifts: $Y_3^{\rightarrow}$ and $Y_3^{\leftarrow}$

By Theorem 4, every d-dimensional effective subshift can be lifted into a $d+1$-dimensional sofic subshift. Using this result, we lift $Y_2^{\rightarrow}$ and $Y_2^{\leftarrow}$ into 3D sofic subshifts $Y_3^{\rightarrow}$ and $Y_3^{\leftarrow}$:

$$Y_3^{\rightarrow} = \{x \in (\Sigma \times \{\rightarrow, \leftarrow\} \times \{\blacksquare, \square\})^{\mathbb{Z}^3} : \exists x' \in Y_2^{\rightarrow}, \forall i, k \in \mathbb{Z}, \forall j' \in \mathbb{Z}, x_{i,j',k} = x'_{i,k}\}$$
$$Y_3^{\leftarrow} = \{x \in (\Sigma \times \{\rightarrow, \leftarrow\} \times \{\blacksquare, \square\})^{\mathbb{Z}^3} : \exists x' \in Y_2^{\leftarrow}, \forall i, j \in \mathbb{Z}, \forall k' \in \mathbb{Z}, x_{i,j,k'} = x'_{i,j}\}$$

Note that the lifts are not made along the same coordinates: a position with $\blacksquare$ in $Y_2^{\rightarrow}$ lifts into a line directed by $(0,1,0)$ in $Y_3^{\rightarrow}$, and a position with $\blacksquare$ in $Y_2^{\leftarrow}$ lifts into a line directed by $(0,0,1)$ in $Y_3^{\leftarrow}$.

Sofic 3D subshift: Y_3

We obtain Y_3 by "fusing" the two previous subshifts. Formally,

$$Y_3 = \{x \in (\Sigma \times \{\rightarrow, \leftarrow\} \times \{\blacksquare, \square\} \times \{\blacksquare, \square\})^{\mathbb{Z}^3} : \pi_{1,2,3}(x) \in Y_3^{\rightarrow} \text{ and } \pi_{1,2,4}(x) \in Y_3^{\leftarrow}\}.$$

Since $Y_3^{\rightarrow}$ and $Y_3^{\leftarrow}$ are sofic, their cartesian product $Y_3^{\rightarrow} \times Y_3^{\leftarrow}$ is also sofic. Y_3 is the projection on Layers $1,2,3,6$ of $Y_3^{\rightarrow} \times Y_3^{\leftarrow}$ with the additional local condition that the first two layers coincide (i.e. $\pi_{1,2}(x) = \pi_{4,5}(x)$), so it is sofic as well.

▷ Claim 15. A configuration of Y_3:

1. breaks every periodicity vector $(n, \cdot, \cdot)$ for $n \geq 1$.
2. every slice $(i, \cdot, \cdot)$ containing $(q_0, \rightarrow)$ on the first two layers and corresponding to the lift of a single position of level $\ell \in [0, +\infty]$ in $T_{\mathcal{M}}$, is periodic with periods $(0,1,0)$ and $(0,0,2^\ell)$ but breaks every period $(\cdot, \cdot, n)$ for $1 \leq n < 2^\ell$. The same is true with $(q_0, \leftarrow)$ with vectors $(0, 2^\ell, 0)$ and $(0,0,1)$.

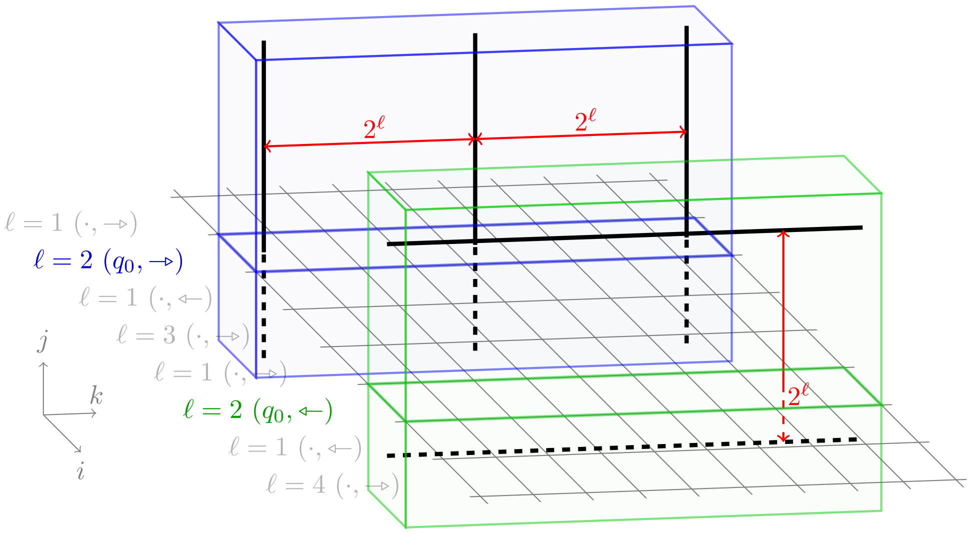


Figure 1 A configuration of Y_3. To the left of each slice $(i,\cdot,\cdot)$ is its level ℓ and the values on Layers 1 and 2. We highlight two slices of level 2: at the front, marked by $(q_0, \leftarrow)$ with horizontal lines; at the back, marked by $(q_0, \rightarrow)$ with vertical lines.

Proof. **1.** the Toeplitzification of alternating $\rightarrow$ and $\leftarrow$ is aperiodic, so Layer 2 breaks all vectors $(n,\cdot,\cdot)$ for $n \geq 1$.

2. Layer 1 and 2 are lifted along the last two dimensions, so they cannot break any such vectors. Layer 3 is lifted along the second dimension so it is $(0,1,0)$-periodic, and breaks the required vectors from the last condition in the definition of $Y_2^{\rightarrow}$. Layer 4 is $\square$ everywhere since it is not marked by $(q_0, \leftarrow)$. $\triangleleft$

3.3 Proof of the reduction RS $\leq$ AD

▶ **Lemma 16.** *A configuration in Y_3 is aperiodic if, and only if, it corresponds to a run of $\mathcal{M}$ in which q_0 occurs infinitely often.*

Proof. Using Claim 15,

- Let $y \in Y_3$ be a configuration corresponding to a run of $\mathcal{M}$ that visit q_0 infinitely often.
  - If $(q_0, \rightarrow)$ appears at a level ℓ, all vectors $(0,\cdot,n)$ for $1 \leq n < 2^\ell$ are broken on Layer 3;
  - Similarly for $(q_0, \leftarrow)$ and vectors $(0,n,\cdot)$ on Layer 4.

  Therefore all vectors $(0,\cdot,\cdot)$ are broken at some level, and y is an aperiodic configuration.
- Let $y \in Y_3$ be a configuration corresponding to a run of $\mathcal{M}$ that does not visit q_0 after some time $N \in \mathbb{N}$. Let $a_\infty \in \Sigma \times \{\rightarrow, \leftarrow\}$ be the value on Layers 1 and 2 of the *single* position of infinite level in z, if it exists.
  - If $a_\infty \neq (q_0, \leftarrow)$, positions marked by $(q_0, \leftarrow)$ must be of level $\leq N$, so y is periodic of period $(0, 2^N, 0)$.
  - Similarly, if $a_\infty \neq (q_0, \rightarrow)$, then y is periodic of period $(0, 0, 2^N)$.

  All in all, y is not aperiodic. ◀

4 Σ_1^1-completeness for $\mathbb{Z}^4$ SFTs

▶ **Theorem 17.** ***AD** for $\mathbb{Z}^4$ SFTs is a Σ_1^1-complete problem.*

Note that this implies the same result for sofic and effective subshifts, and also for any dimension $d \geq 4$. Indeed, in the construction X_4 below, there are two dimensions which are always aperiodic, and two others which may or may not be periodic. Let A be the $\mathbb{Z}^d$ lift of any $\mathbb{Z}^{d-2}$ aperiodic SFT ($d-2 \geq 2$), and X_d the $\mathbb{Z}^d$ lift of X_4. Then the cartesian product $A \times X_d$ is aperiodic if and only if X_4 is.

4.1 Outline of the proof

This proof has the same structure as Theorem 11 with some adaptations for $\mathbb{Z}^4$ SFTs. We reduce to the problem **SR**: given $\mathcal{M}$ and q_0, we create an SFT X_4 that contains an aperiodic configuration if and only if $\mathcal{M}$ admits a run from the empty word which visits q_0 infinitely often. To do this, we use repeated lines along two dimensions (3 and 4) to break all periods up to a length controlled by a computation embedded in the configuration.

1. In Section 4.2, we build X_T^2, a $\mathbb{Z}^2$ version of the Toeplitz structure X_T;
2. In Section 4.3, we build auxiliary SFTs $X_3^{\rightarrow}$ and $X_3^{\leftarrow}$ (counterparts to $Y_2^{\rightarrow}$ and $Y_2^{\leftarrow}$);
3. In Section 4.4, we build X_4 and prove the reduction.

The main difference is that the finite type case requires an additional dimension to embed computations and some construction lines (dimensions 1 and 2). Remember that the subshift $T_{\mathcal{M}}$ of $\mathbb{Z}$ Toeplitzified sequences of states of runs of $\mathcal{M}$ is effective ; instead, we define a $\mathbb{Z}^2$ version $T^2_{\mathcal{M}}$ that is sofic and aperiodic, and the projection of a $\mathbb{Z}^2$ aperiodic SFT. We then need two additional dimensions in which the subshift can be periodic or aperiodic, since the position of infinite level can break period along one dimension uncontrollably.

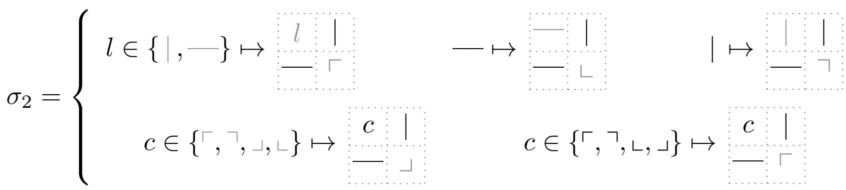
Furthermore, to control the length of the vectors being broken, we need to measure distances between lines (as in Y_3). With SFTs, copying a distance from one dimension to another can only be done with diagonals. Therefore, instead of *lines* to break periods, we use a more complex *diagonal SFT D* that we embed in slices $(i, \cdot, \cdot, \cdot)$ only on dimensions 2 and 3 (on symbols $\rightarrow$) or 2 and 4 (on symbols $\leftarrow$). This way, the computation embedded in the first two dimensions can control the length of the broken periodicity vectors.

4.2 $T^2_{\mathcal{M}}$: $\mathbb{Z}^2$ Toeplitz corresponding to state sequences of $\mathcal{M}$

Binary Toeplitz structure

In this section, we use a $\mathbb{Z}^2$ subshift X_T^2 on the alphabet $\{⌜, ⌝, ⌞, ⌟, |, —\}$ whose structure is a two-dimensional analog of X_T.

It is defined by the substitution σ_2 on the alphabet $\{⌜, ⌝, ⌞, ⌟, |, —, ⌜, ⌝, ⌞, ⌟, |, —\}$:



As before, X_{σ_2} is the subshift whose forbidden patterns are all the patterns which do not appear in the configuration $\sigma_2^{\omega}(⌜)$ (any other seed symbol would do).

▶ **Definition 18** (binary bi-Toeplitz structure). *X_T^2 is the color-forgetting projection of X_{σ_2} on the alphabet $\{⌜, ⌝, ⌞, ⌟, |, —\}$.*

X_T^2 is a sofic subshift, as it is the projection of a Robinson tiling (see [16]).

In a configuration of X_T^2, ignoring symbols $|$ and —:

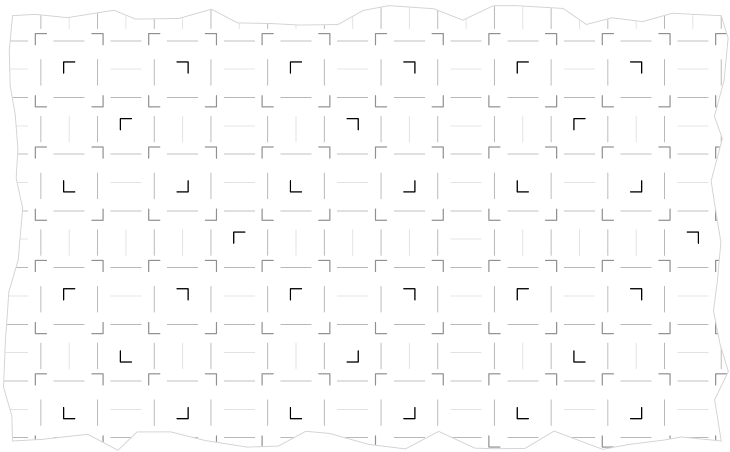


Figure 2 A configuration of X_{σ_2}.

1. corner symbols $\{\ulcorner, \urcorner, \llcorner, \lrcorner\}$ can be grouped together to form squares. A square is of level n if its edges have length 2^n. There may exist either one or four squares of infinite size, whose level is said to be infinite;
2. each line only contains symbols in either $\{\ulcorner, \urcorner\}$ or $\{\llcorner, \lrcorner\}$, all of the same level. If a line does not contain any corner, its level is said to be infinite;
3. the vertical distance between two consecutive lines of the same level ℓ is 2^ℓ, and those lines contain the same symbols.

The corresponding statements hold for columns.

$T^2_{\mathcal{M}}$: $\mathbb{Z}^2$ Toeplitzification of sequences of states of $\mathcal{M}$

▶ **Definition 19** ($\mathbb{Z}^2$ Toeplitzification of a set of sequences)**.** *Given a set of sequences $A \subseteq \Sigma^{\mathbb{N}}$, we define the corresponding $\mathbb{Z}^2$ Toeplitzified subshift T^2_A on the alphabet $\Sigma_T = \Sigma \times \{\rightarrow, \leftarrow\} \times \{\ulcorner, \llcorner, \urcorner, \lrcorner, |, —\}$ as:*

$$T^2_A = \left\{ x \in (\Sigma_T)^{\mathbb{Z}^2} : \begin{array}{l} \pi_{1,2}(x) \in (T_A)^{\uparrow}, \pi_3(x) \in X^2_T \\ \forall i,j \in \mathbb{Z}, \begin{array}{l} \pi_3(x_{i,j}) \in \{\ulcorner, \llcorner\} \implies \pi_2(x_{i,j}) = \rightarrow \\ \pi_3(x_{i,j}) \in \{\urcorner, \lrcorner\} \implies \pi_2(x_{i,j}) = \leftarrow \end{array} \end{array} \right\}$$

In other words, T^2_A superimposes the structure of a $\mathbb{Z}$ Toeplitzification with the $\mathbb{Z}^2$ structure X^2_T we define above. As before, we denote $T^2_{\mathcal{M}} := T^2_{S_{\mathcal{M}}}$ where $S_{\mathcal{M}}$ of the set of sequences of states corresponding to runs of $\mathcal{M}$.

▶ **Lemma 20.** *$T^2_{\mathcal{M}}$ is a sofic subshift.*

Proof.
- Layers 1 and 2 are $\mathbb{Z}^2$ lifts of configurations of $T_{\mathcal{M}}$, which is an effective subshift (Lemma 14). By Theorem 4, Layers 1 and 2 form a sofic subshift;
- Layer 3 is composed of configurations of X^2_T, which is a sofic subshift;
- the additional condition defining $T^2_{\mathcal{M}}$ (synchronizing the content of Layers 2 and 3) is of finite type. ◀

4.3 Auxiliary $\mathbb{Z}^2$ and $\mathbb{Z}^3$ SFTs

The diagonal SFT D

The diagonal SFT D is defined by adjacent matching patterns on the alphabet Σ_D:

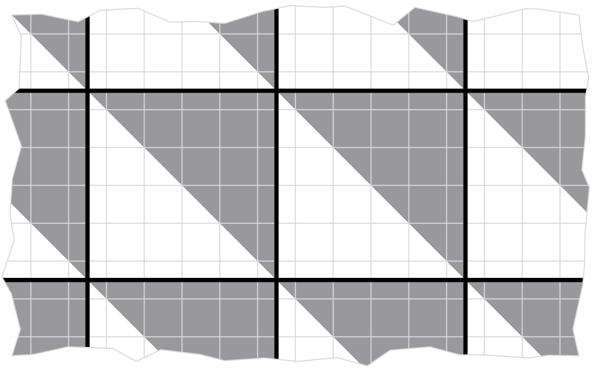
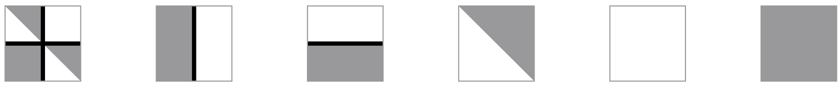


Figure 3 A configuration of D.



As soon as it contains two black lines or columns, a configuration of D is periodic and consists of repeated squares. It also contains aperiodic configurations with infinite squares.

$\mathbb{Z}^3$ SFTs $X_3^{\rightarrow}$ and $X_3^{\leftarrow}$

This section is written for $X_3^{\rightarrow}$; it applies to $X_3^{\leftarrow}$ by flipping the arrows and corners. $T^2_{\mathcal{M}}$ is a sofic subshift, so it is the projection of some $\mathbb{Z}^2$ SFT X_2. To create $X_3^{\rightarrow}$, we lift X_2 then add an additional layer for D:

- Layers 1 to 4: $\mathbb{Z}^3$ lift of X_2;
- Layer 5 : each slice $(i,\cdot,\cdot)$ contains a configuration $d^i \in D$. If the slice has $(q_0, \rightarrow)$ on its first two layers, then the configuration d^i is "synchronized" with the underlying configuration of $T^2_{\mathcal{M}}$ on Layer 3, that is: vertical lines $\rule{1pt}{8pt}$ in d^i only appear on Layer 5 at positions marked by corners $\ulcorner, \llcorner$ on Layer 3. Otherwise, the slice on Layer 5 is left blank.

See Figure 4 for a visual help. Formally, if Σ_X and Σ_D are the alphabets of X_2 and D, $X_3^{\rightarrow}$ can be expressed as:

$$\left\{ x \in (\Sigma_X \times (\Sigma_D \cup \square))^{\mathbb{Z}^3} : \begin{array}{l} \exists y \in X_2,\ \forall i,j,k \in \mathbb{Z},\ \pi_{1,2,3,4}(x_{i,j,k}) = y_{i,j} \\ \forall i \in \mathbb{Z},\ \exists d^i \in D,\ \forall j,k \in \mathbb{Z},\ \pi_5(x_{i,j,k}) = d^i_{j,k} \\ \forall i,j,k \in \mathbb{Z}, \begin{array}{l} \pi_{1,2}(x_{i,j,k}) \neq (q_0, \rightarrow) \implies \pi_5(x_{i,j,k}) = \square \\ \pi_5(x_{i,j,k}) \in \{ \boxplus, \square\!\!\!| \} \iff \pi_3(x_{i,j,k}) \in \{\ulcorner, \llcorner\} \end{array} \end{array} \right\}$$

On every slice marked by $\rightarrow$, the configuration of D breaks all periods smaller than its squares, and the size of the squares is controlled by the level of the slice in $T_{\mathcal{M}}$. Therefore:

▷ Claim 21. A configuration of $X_3^{\rightarrow}$:

1. breaks every periodicity vector $(n,\cdot,\cdot)$ and $(\cdot,n,\cdot)$ for $n \geq 1$.
2. for every slice $(i,\cdot,\cdot)$ containing $(q_0, \rightarrow)$ on the first two layers and corresponding to a column of level ℓ in X^2_T, Layer 5 breaks every vector $(\cdot,\cdot,n)$ for $1 \leq n < 2^\ell$.

Proof. **1.** X_2 is aperiodic because X^2_T is also aperiodic, so all vectors $(n,\cdot,\cdot)$ and $(\cdot,n,\cdot)$ are broken for $n \geq 1$.

2. For a slice $(i,\cdot,\cdot)$ of level ℓ, the distance along $(0,1,0)$ between two consecutive lines in Layer 3 (ie. in X^2_T) is exactly 2^ℓ (see. Section 4.2, point 3 in the list of properties of configurations in X^2_T). So, Layer 5 breaks every smaller period in this direction. ◁

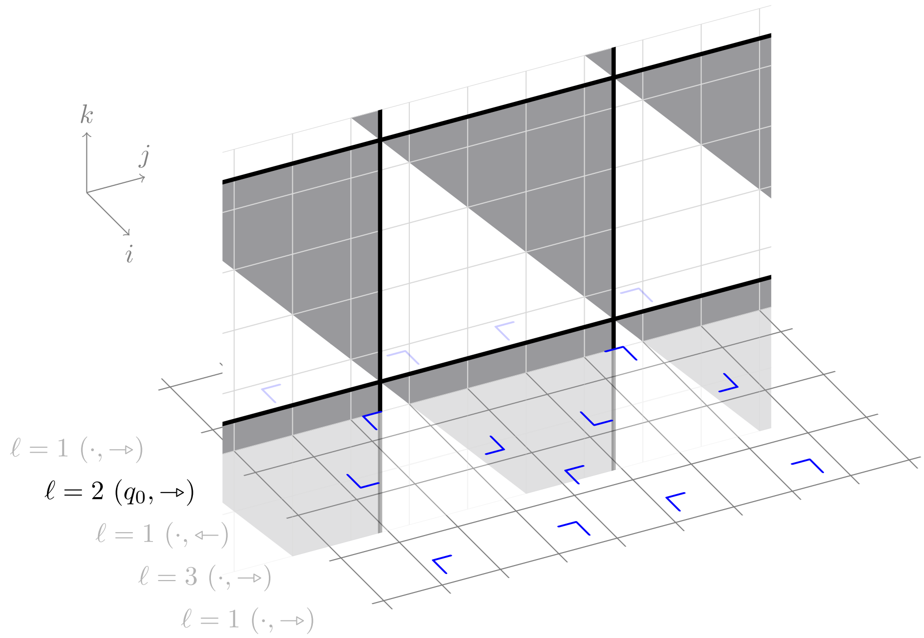


Figure 4 A configuration of $X_3^{\rightarrow}$. The horizontal plane contains a configuration $x \in X_T^2$, and the slice of level 2 marked by $(q_0, \rightarrow)$ contains a configuration of D "synchronised" with the squares of x.

4.4 $\mathbb{Z}^4$ SFT X_4 and proof of the reduction

Creation of X_4

Similarly to Y_3 (Section 3.2), we build X_4 by "fusing together" $X_3^{\rightarrow}$ and $X_3^{\leftarrow}$. Formally:

$$X_4 = \{x \in (\Sigma_X \times T \times T)\}^{\mathbb{Z}^4} : \exists x^{\rightarrow} \in X_3^{\rightarrow}, \exists x^{\leftarrow} \in X_3^{\leftarrow},$$
$$\forall i, j, k, l \in \mathbb{Z}, \pi_{1,2,3,4,5}(x_{i,j,k,l}) = x^{\rightarrow}_{i,j,l} \text{ and } \pi_{1,2,3,4,6}(x_{i,j,k,l}) = x^{\leftarrow}_{i,j,k}\}$$

▷ Claim 22. X_4 is an SFT.

Proof. Both $X_3^{\rightarrow}$ and $X_3^{\leftarrow}$ are SFTs, since X_2 is an SFT. ◁

Reduction RS $\leq$ AD

▶ **Lemma 23.** *There exists an aperiodic configuration in X_4 if and only if there exists a run of $\mathcal{M}$ in which q_0 occurs infinitely often.*

Proof. This is the same proof as for Lemma 16, except that vectors along the first two dimensions are broken by the Toeplitz structure on layer 3. Otherwise, Layers 5 and 6 break every vector $(0, 0, \cdot, \cdot)$ if and only if the run visits q_0 infinitely often. ◀

This concludes the proof of Theorem 17.

5 Complexity and aperiodic configurations

The *complexity function* of a $\mathbb{Z}^d$ subshift X is $N_X(n) = \#\{w \in \Sigma^{n^d} : \exists x \in X, w \sqsubseteq x\}$.

Intuitively, subshifts of high complexity are expected to have aperiodic configurations. For example, [7, Theorem 10] proves that a $\mathbb{Z}^2$ subshift with no aperiodic configurations is almost topologically conjugated to (i.e. "nearly behaves as") a $\mathbb{Z}$ subshift of the same type. In particular, its complexity function is at most linearly exponential.

▶ **Definition 24** (Dimensional entropy). *The* entropy of dimension d $h_d(X)$ *is defined as:*

$$h_d(X) = \limsup_{n\to+\infty} \frac{\log N_X(n)}{n^d} \in [0, +\infty]$$

[10, Corollary 13] entails that $h_d(X) > 0$ for an $\mathbb{Z}^d$ SFT implies the existence of aperiodic configurations. We improve this result as follows:

▶ **Proposition 25.** *Let X be a $\mathbb{Z}^d$ SFT or sofic subshift. If $h_{d-1}(X) = +\infty$, then there exists an aperiodic configuration in X.*

Proof. W.l.o.g., we assume X is a SFT defined by adjacency constraints. $N_b(X, n)$, the number of boundaries of admissible cubes of size n^d, is at most $\Sigma^{2dn^{d-1}}$. $\log N_X(n)/n^{d-1}$ is unbounded, so there exists some n such that:

$$2d < \frac{\log N_X(n)}{n^{d-1}} \iff \Sigma^{2dn^{d-1}} < N_X(n) \implies N_b(X, n) < N_X(n)$$

By the pigeonhole principle, there exists an admissible boundary b of a cube of size n^d that can be filled in at least two admissible ways. Consider a configuration x in which b appears. If x is not already aperiodic, swapping the interior of b at a single arbitrary position leads to a configuration $x' \in X$ which is aperiodic. ◀

Proposition 25 is tight: the $\mathbb{Z}^d$-lift of a $\mathbb{Z}^{d-1}$ SFT ($d \geq 3$) is periodic by definition, and can have arbitrarily high entropy of dimension $(d-1)$. Proposition 25 shows that **AD** is a problem that is only relevant for low complexity subshifts, which is where its full computational complexity "lies". Indeed, the problem of deciding whether $h_{d-1}(X) = +\infty$ is Π^0_3, which is much easier than the Σ^1_1-completeness of **AD** on $\mathbb{Z}^3$ sofic subshifts.

6 Open problems

The main remaining question is, of course, the case of $\mathbb{Z}^3$ SFTs. The method we developed above to prove Σ^1_1-completeness in the case of $\mathbb{Z}^3$ sofic subshifts cannot apply. Indeed, embedding computations in an SFT requires at least two aperiodic dimensions; and we need two other dimensions (because of the positions of level ∞) which can be periodic or aperiodic.

We conjecture that aperiodic configurations in $\mathbb{Z}^3$ SFTs behave similarly as in $\mathbb{Z}^2$ subshifts: each $\mathbb{Z}^3$ SFT containing aperiodic configurations seems to have "centers" of aperiodicity, i.e. concentric zones in which periods are broken. The distance from the center might depend on the length of the vector, $|\Sigma|$ and the size of the largest forbidden pattern.

However, there seem to be important differences. First, for $\mathbb{Z}^3$-SFT, not all aperiodic configurations have a center of aperiodicity in their orbit closure: this center may be in an unrelated aperiodic configuration. Second, results of [7] are valid for all $\mathbb{Z}^2$ subshifts, but our conjecture must be specific to $\mathbb{Z}^3$ SFTs, and a proof requires SFT-specific techniques.

Considering subshifts on more general groups (other than $\mathbb{Z}^d$), there is an active research theme looking for conditions on groups which make the Domino problem undecidable. In this context, we would like to obtain conditions that make **AD** Π^0_1- or Σ^1_1-complete.

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