LACL - Créteil (France)
ENS Paris-Saclay (France)

Surface entropies of subshifts of finite type

Antonin Callard & Pascal Vanier

Journées SDA2 - 2020

Shifts

Definition 1 2D Shifts

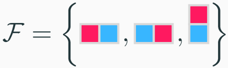
A 2D-*subshift* is a set of colorings $\mathbb{Z}^2 \mapsto \Sigma$ that do not contain some family of forbidden patterns $\mathcal{F}$. Each family of forbidden patterns defines a subshift:

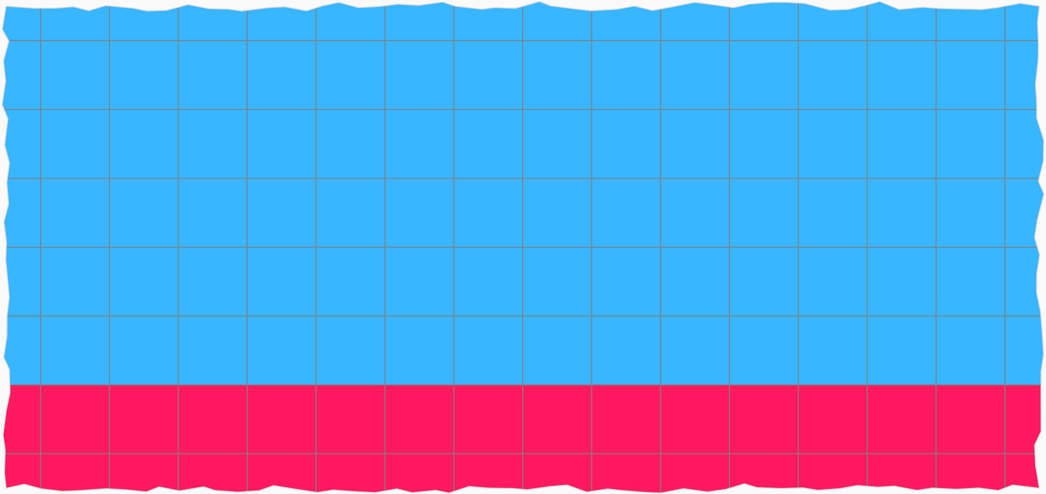
$$X_{\mathcal{F}} = \left\{ x \in \Sigma^{\mathbb{Z}^2} : \forall p \in \mathcal{F}, p \text{ does not appear in } x \right\}$$

Definition 2 Classification of shifts

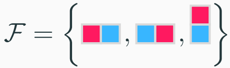
1. A *subshift of finite type* (or SFT) is a subshift that can be defined by a finite family of forbidden patterns.
2. An *effective subshift* is a subshift that can be defined by a recursively enumerable family of forbidden patterns.

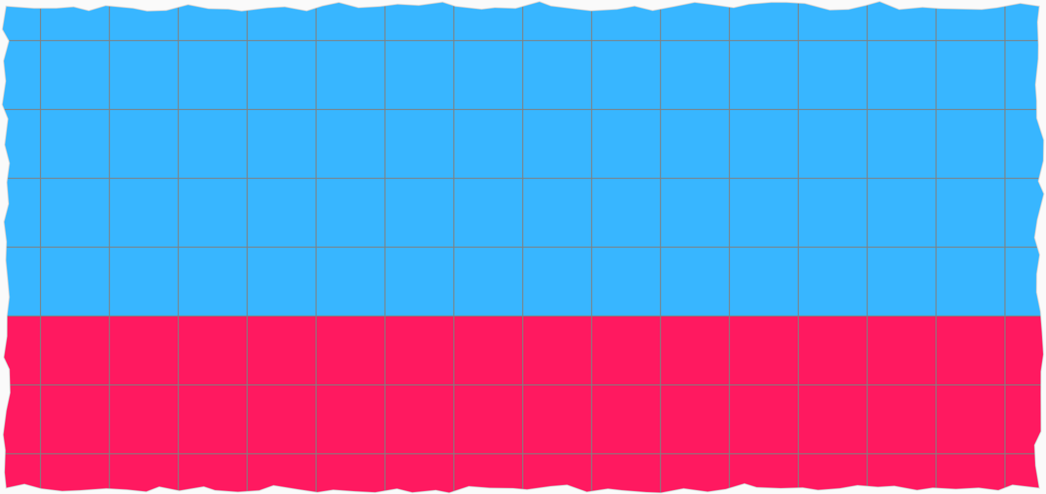
Example



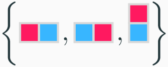


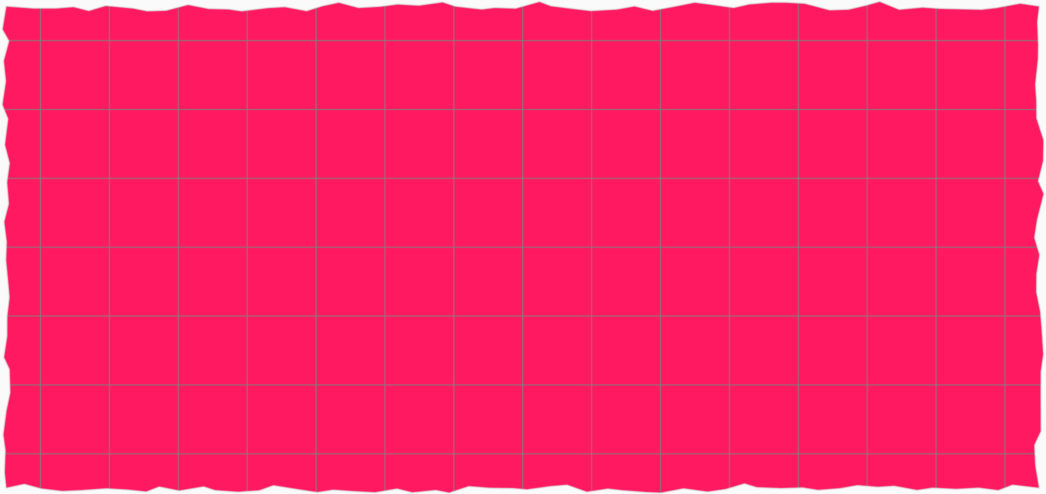
Example



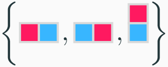


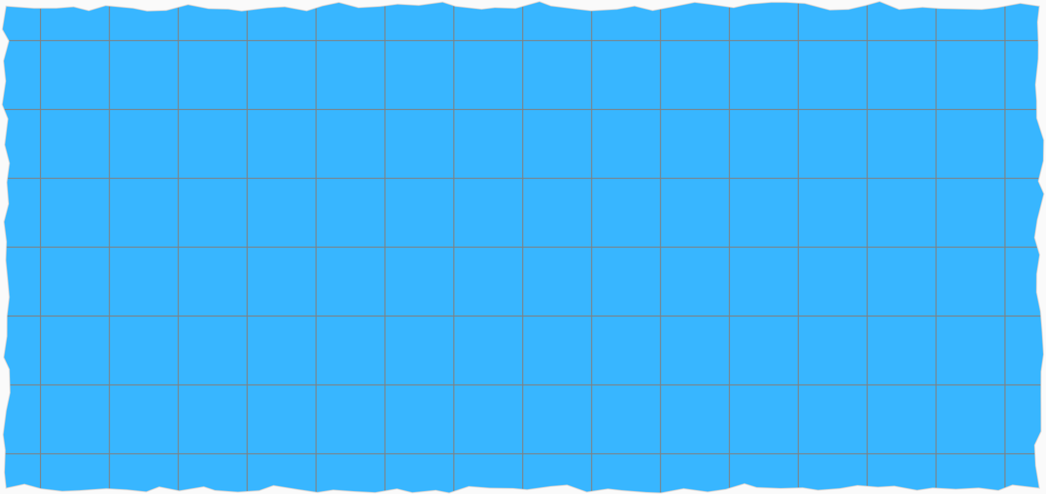
Example

$\mathcal{F} =$ 



Example

$\mathcal{F} =$ 



Definition 3 Complexity function

The *complexity function* $N_n(X)$ of a shift X, for $n \in \mathbb{N}$, is defined as the number of different patterns of size $n \times n$ that appear in X.

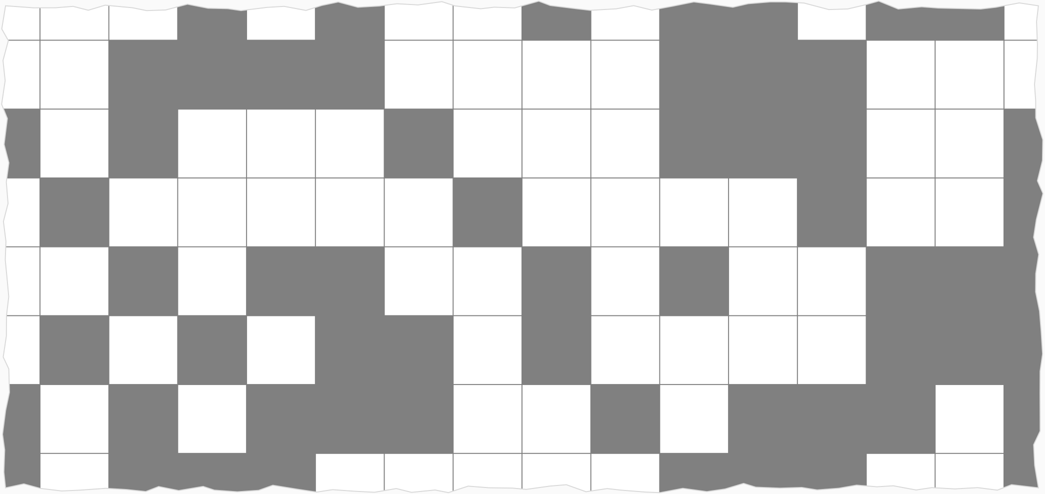
$$\log N_n(X) \simeq hn^2 + h'n + \dots$$

QUESTION:

For an SFT, what are the possible values for h' ?

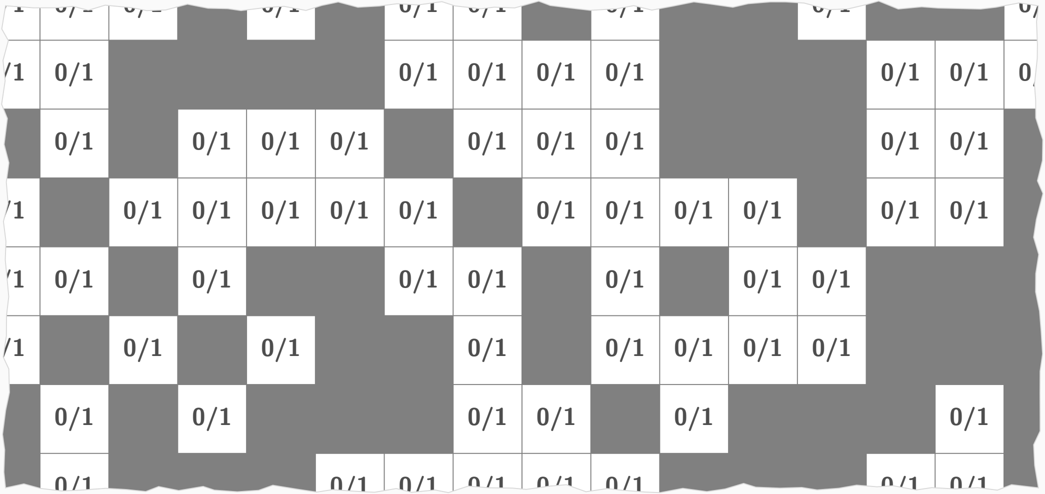
Complexity function is density

$$N_n = \# \text{ Different patterns of size } n \times n$$



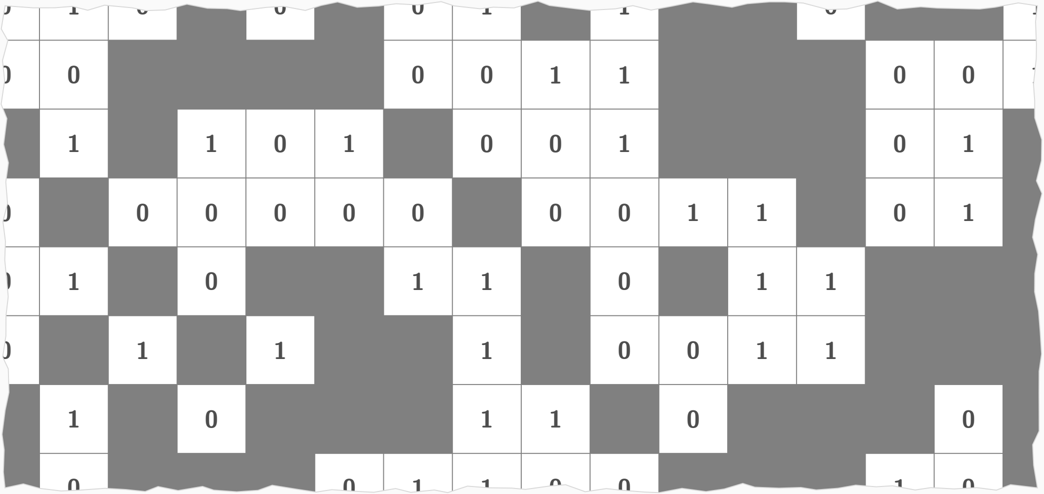
Complexity function is density

$$N_n = \#\ \text{Different patterns of size } n \times n$$



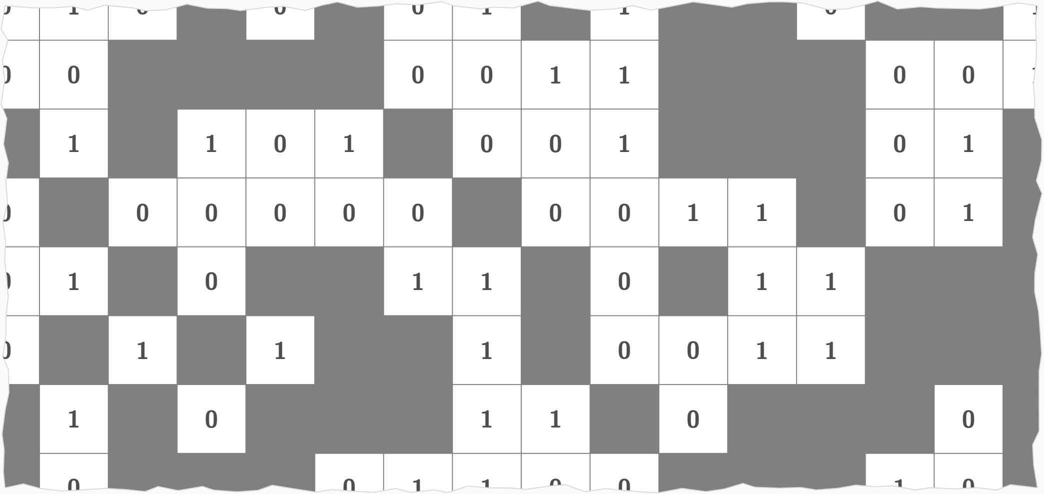
Complexity function is density

$$N_n = \#\ \text{Different patterns of size } n \times n$$



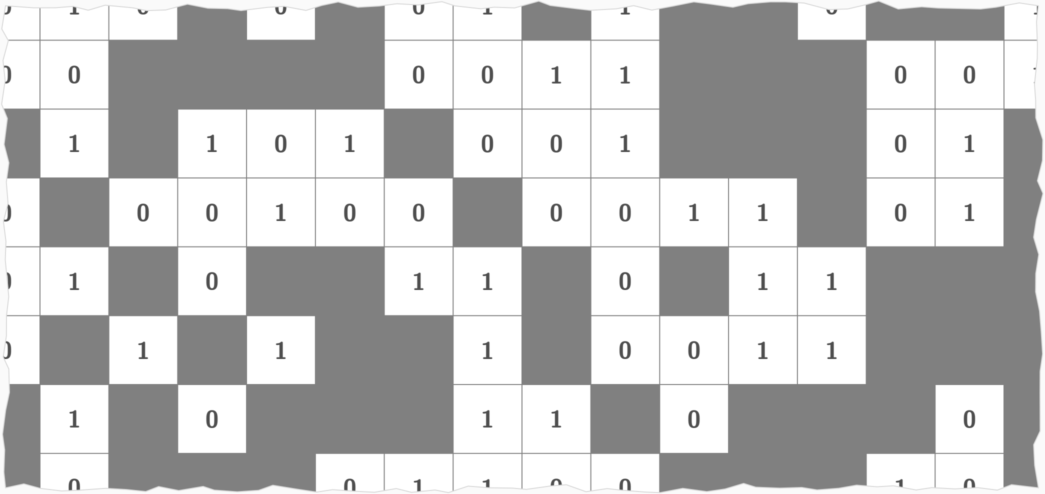
Complexity function is density

$$N_n = \#\ \text{Different patterns of size } n \times n$$



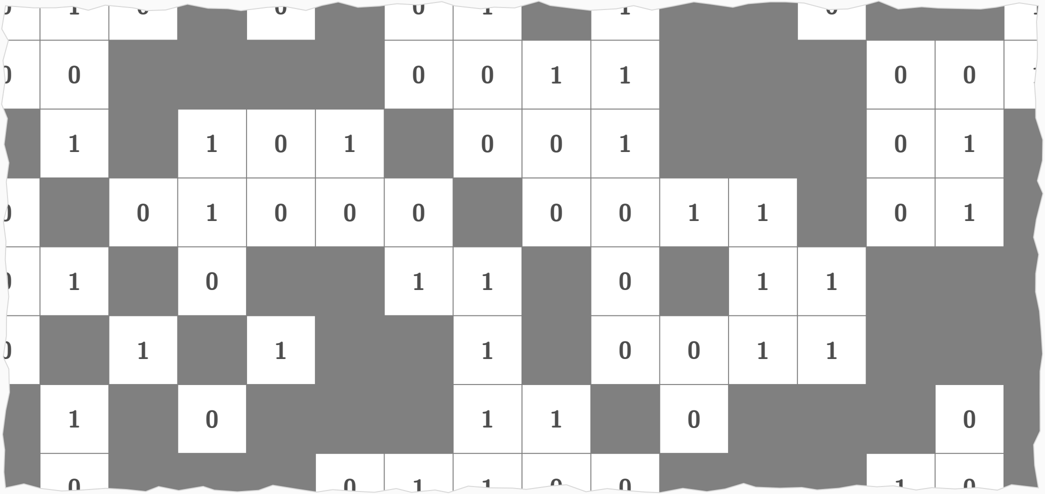
Complexity function is density

$N_n = \#$ Different patterns of size $n \times n$



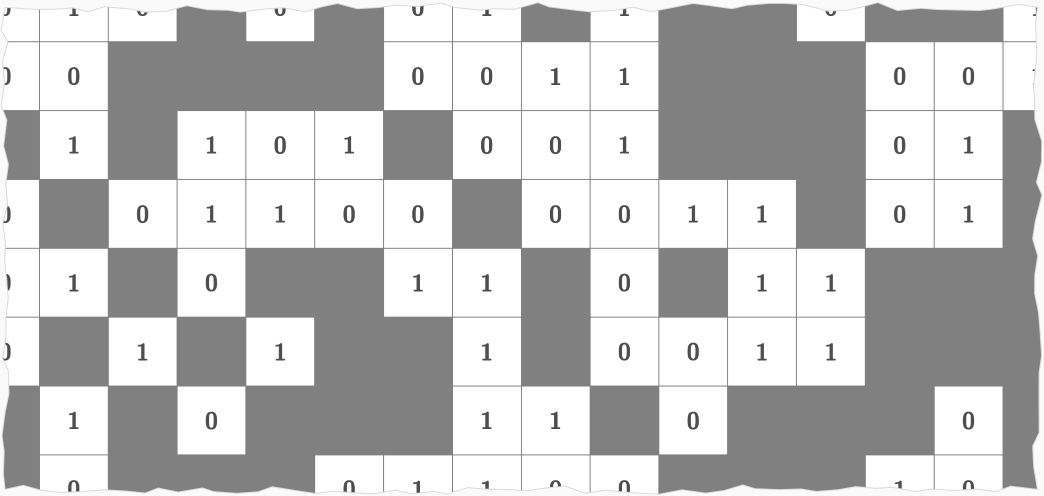
Complexity function is density

$$N_n = \#\ \text{Different patterns of size } n \times n$$



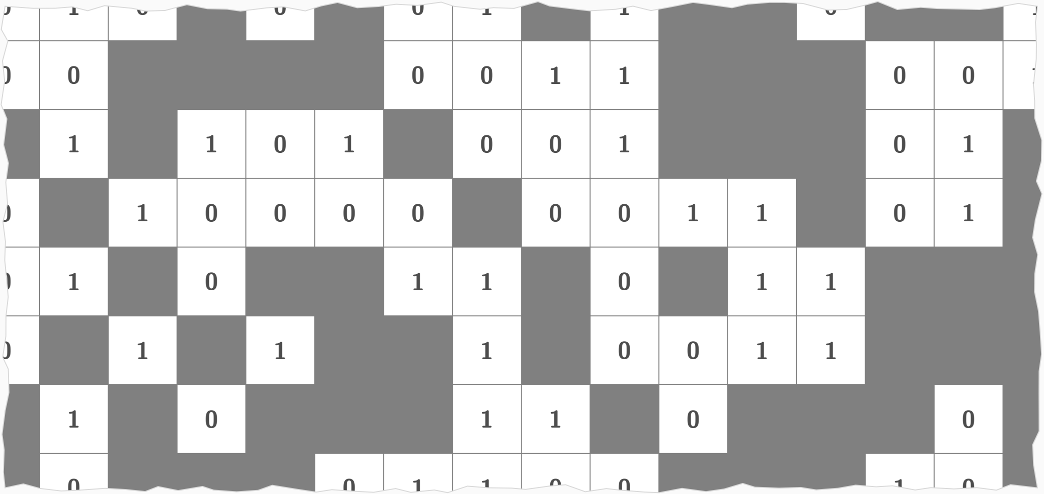
Complexity function is density

$$N_n = \#\ \text{Different patterns of size } n \times n$$



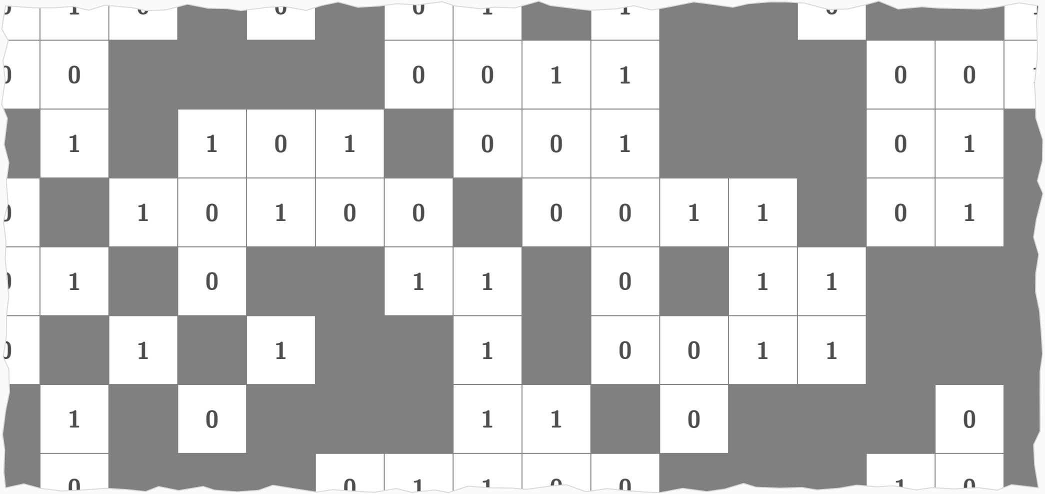
Complexity function is density

$N_n = \#$ Different patterns of size $n \times n$



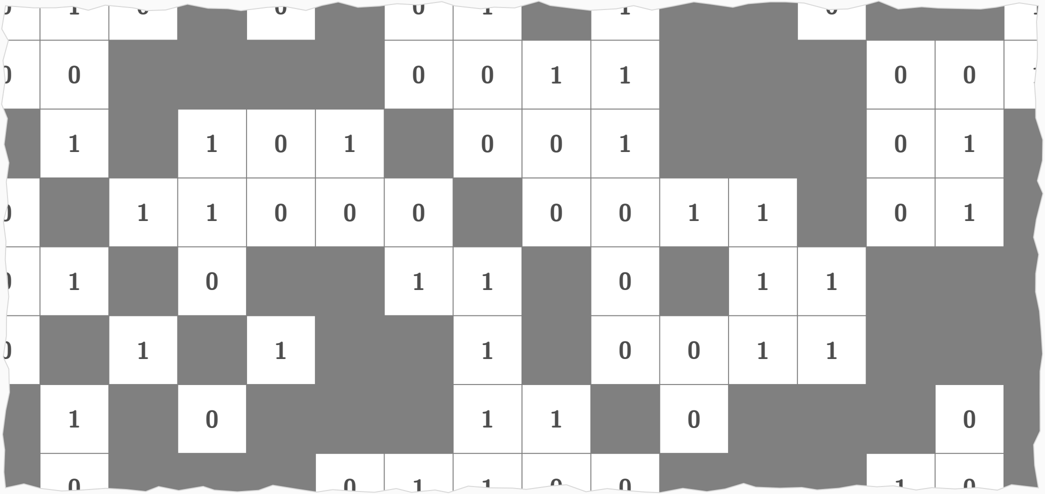
Complexity function is density

$$N_n = \#\ \text{Different patterns of size } n \times n$$



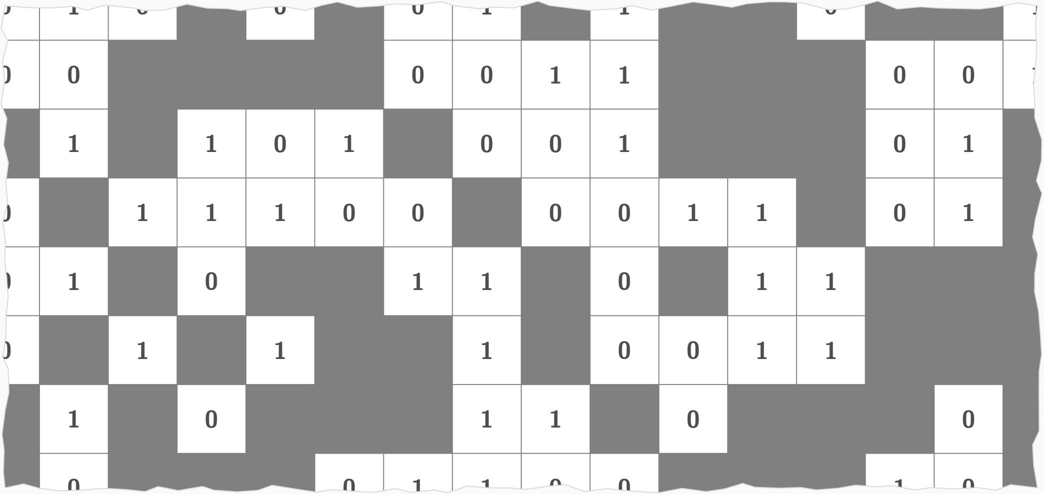
Complexity function is density

$$N_n = \#\ \text{Different patterns of size}\ n \times n$$



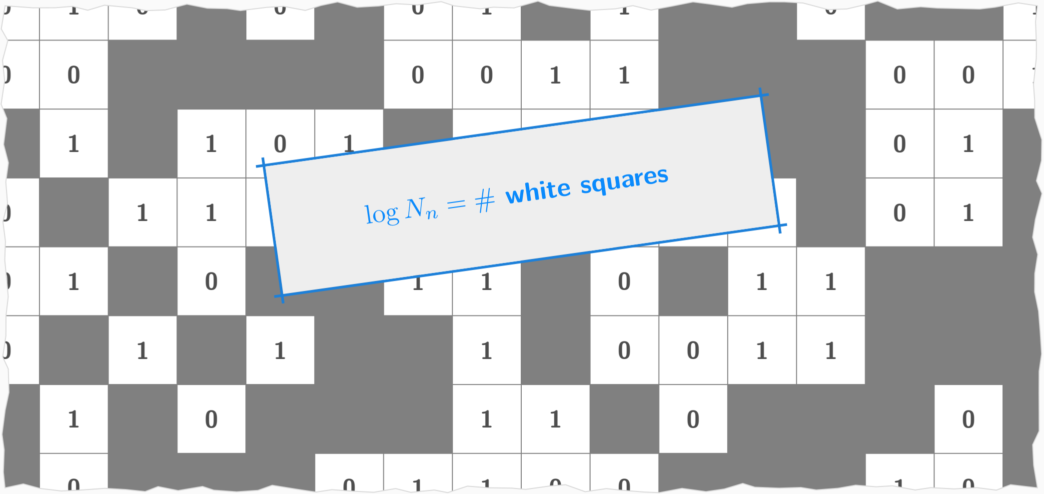
Complexity function is density

$$N_n = \#\ \text{Different patterns of size } n \times n$$



Complexity function is density

$N_n = \#$ Different patterns of size $n \times n$



Topological entropy

What about topological entropies? (Part 1)

$$\log N_n \simeq hn^2$$

[Hochman, Meyerovitch, 2010] proved that topological entropy were the Π_1 real numbers.

$\Longrightarrow$: $h_{\text{top}} \in \Pi_1$.

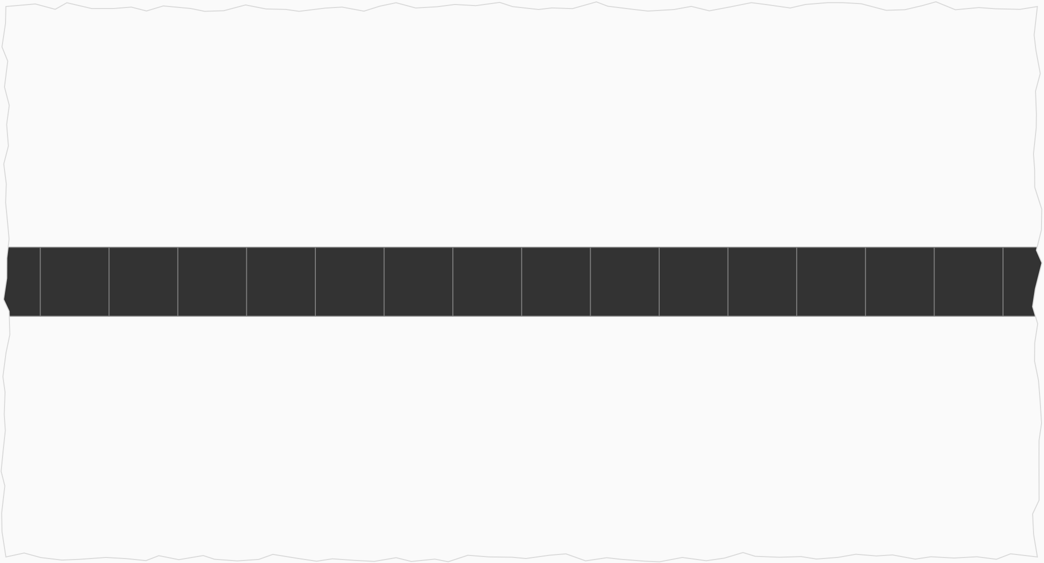
$\Longleftarrow$: Let $h \in \Pi_1$. We build an SFT with topological entropy h.

Arithmetical hierarchy of real numbers

$x \in \Pi_1$ if there exists a recursively enumerable $(r_k)_{k\in\mathbb{N}}$ in $\mathbb{Q}$ such that:

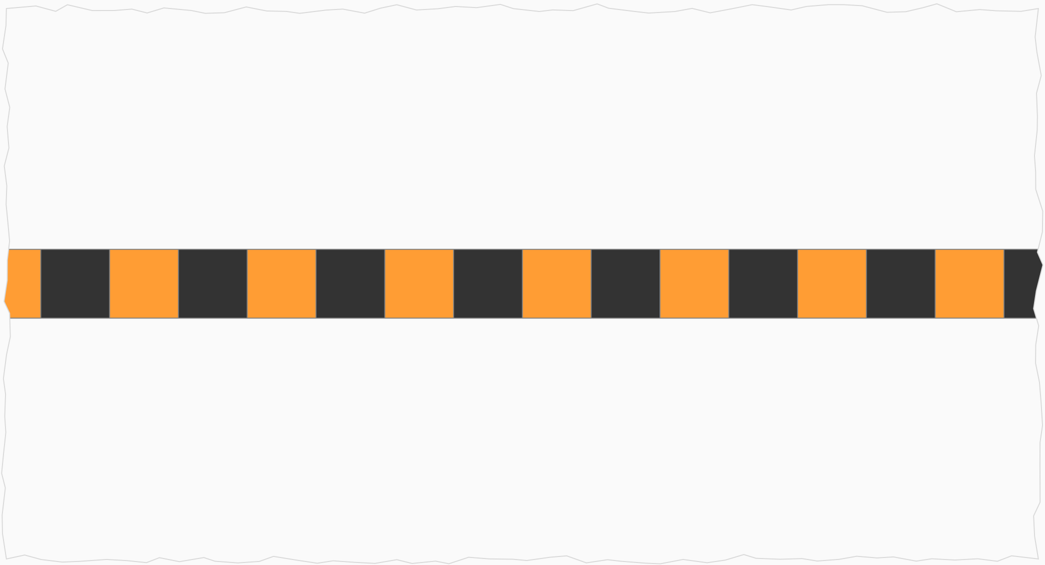
$$x = \inf_k r_k$$

What about topological entropies? (Part 2)



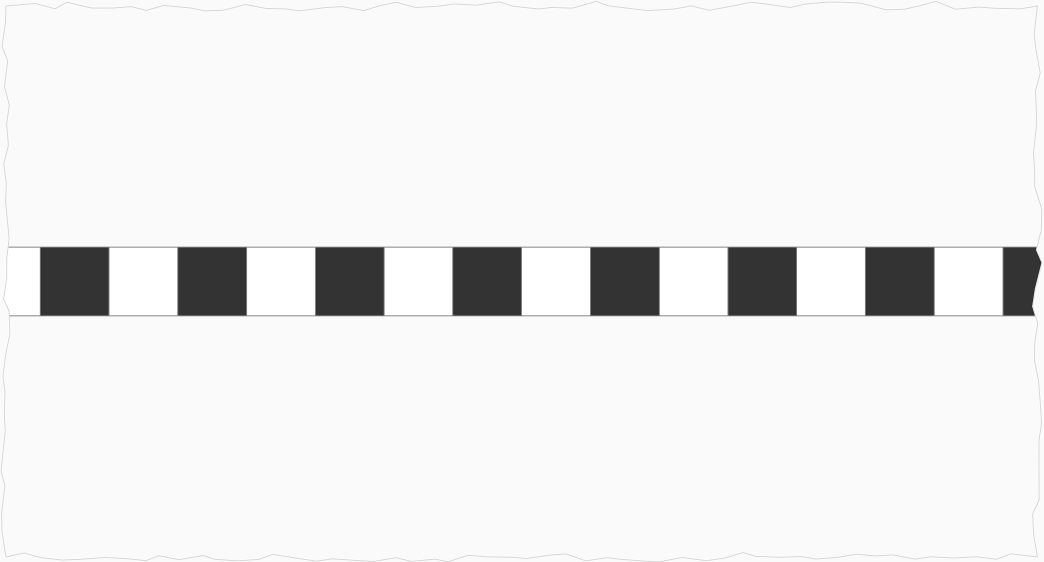
$$h = .10100\ldots$$

What about topological entropies? (Part 2)



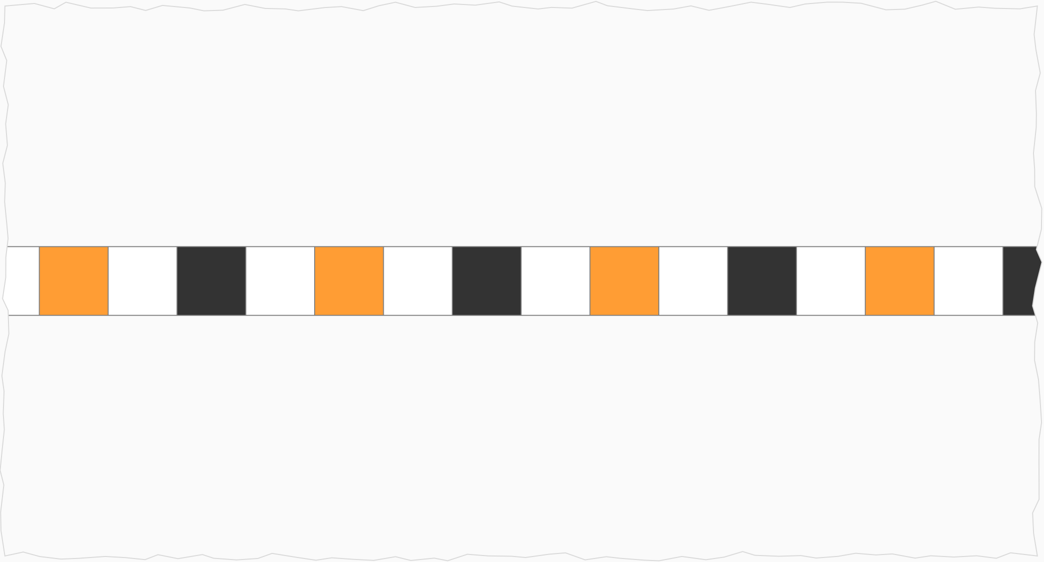
$$h = .10100\ldots$$

What about topological entropies? (Part 2)



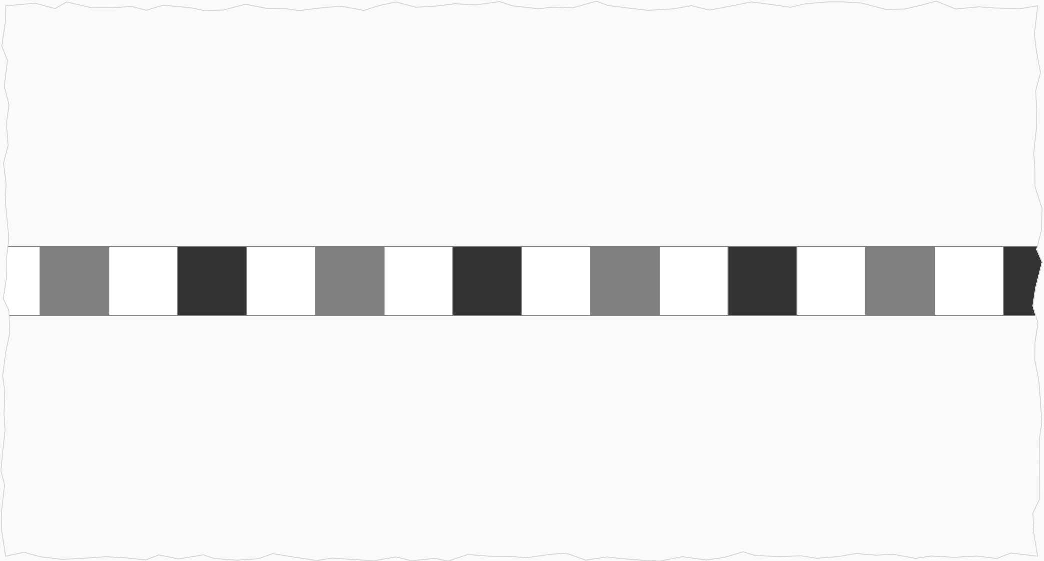
$$h = .10100\ldots$$

What about topological entropies? (Part 2)



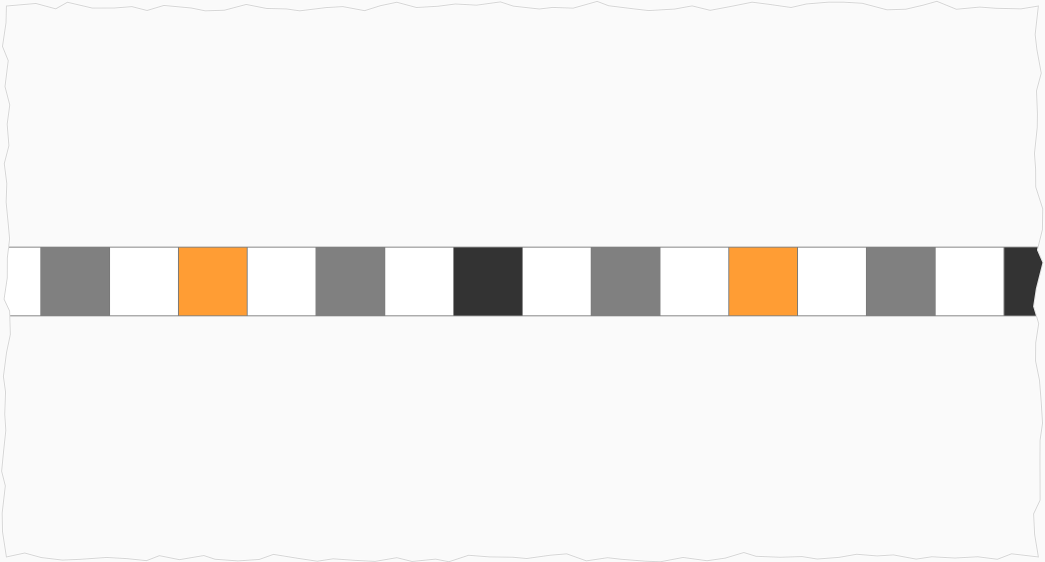
$h = .10100\ldots$

What about topological entropies? (Part 2)



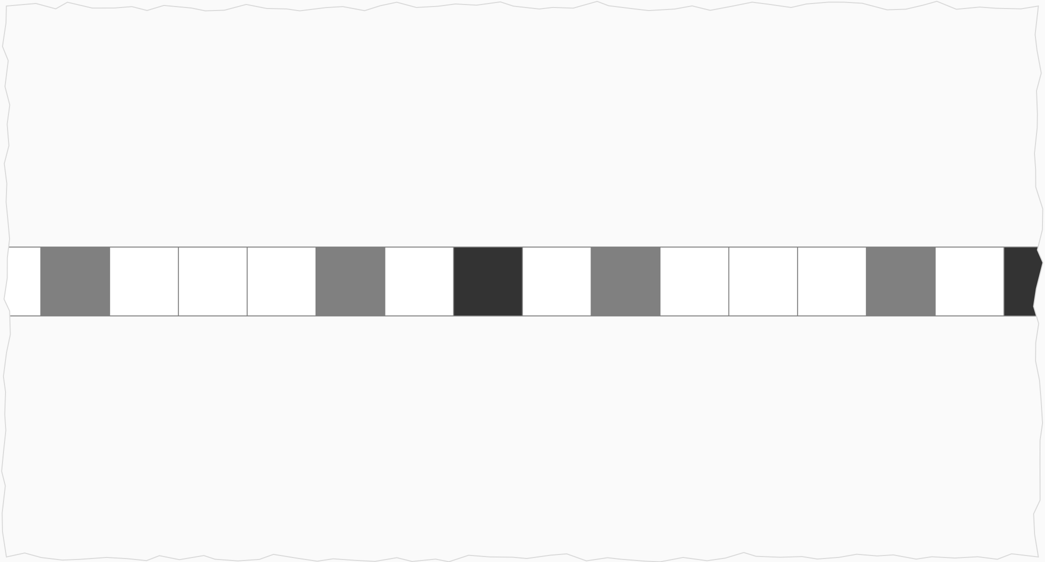
$h = .10100\ldots$

What about topological entropies? (Part 2)



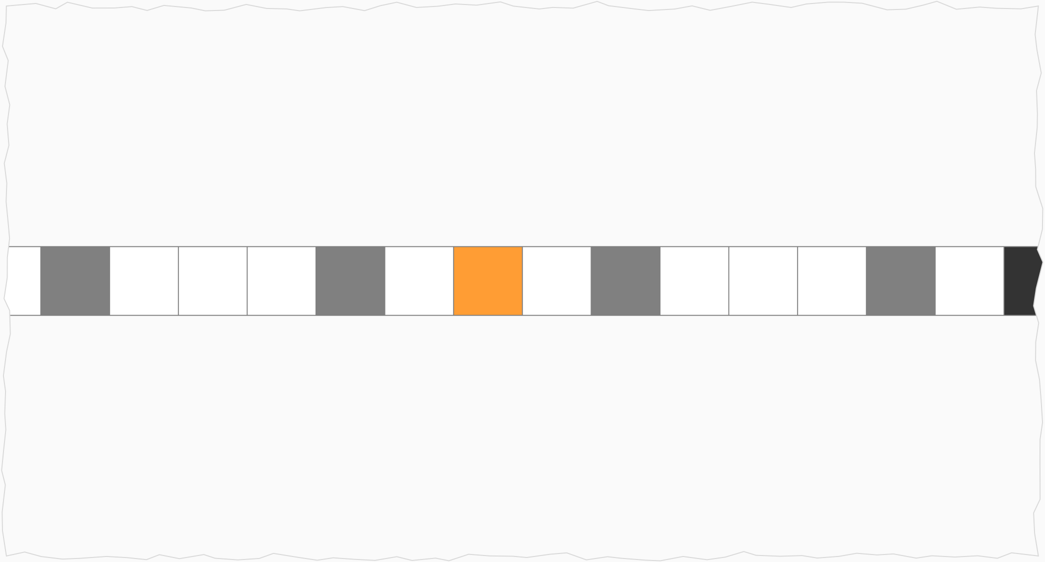
$$h = .10{\color{orange}1}00\ldots$$

What about topological entropies? (Part 2)



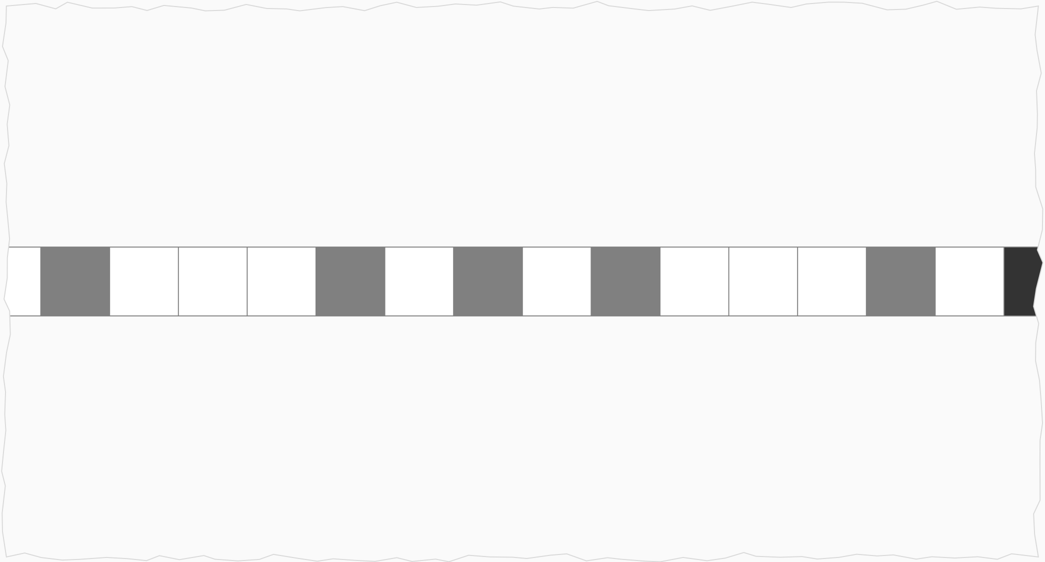
$h = .10100...$

What about topological entropies? (Part 2)



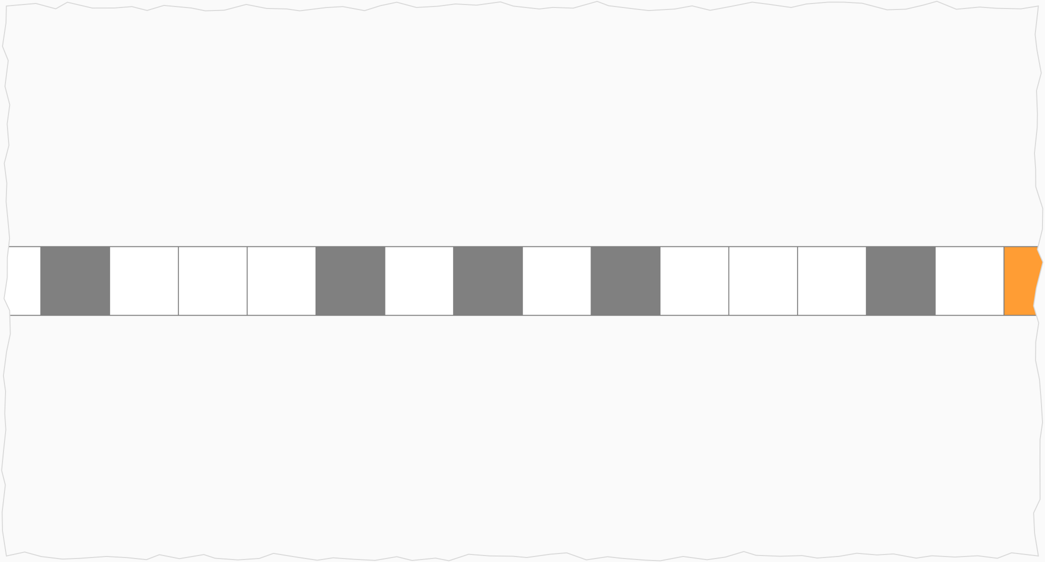
$h = .10100\ldots$

What about topological entropies? (Part 2)



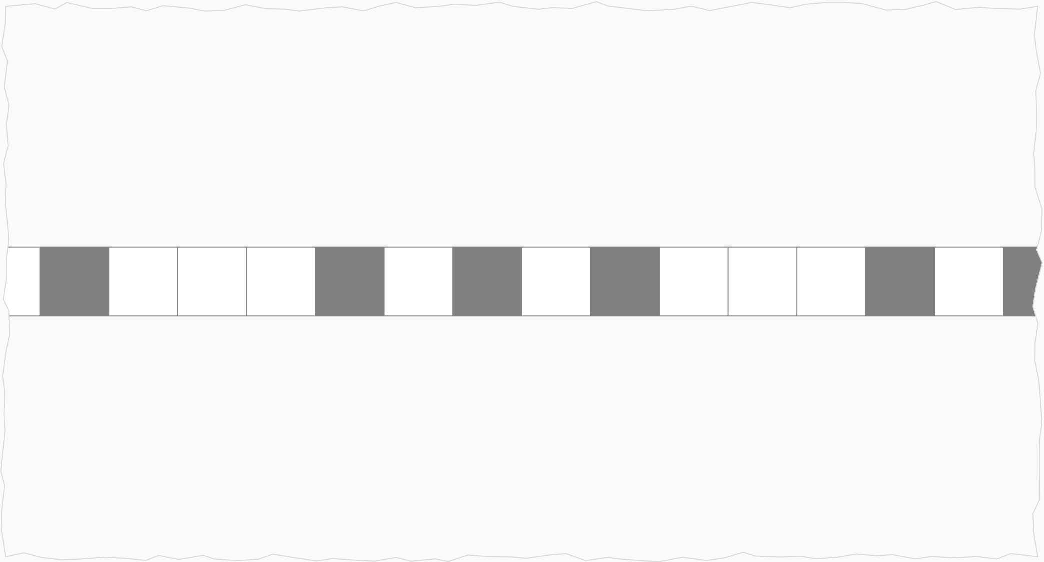
$h = .10100\ldots$

What about topological entropies? (Part 2)



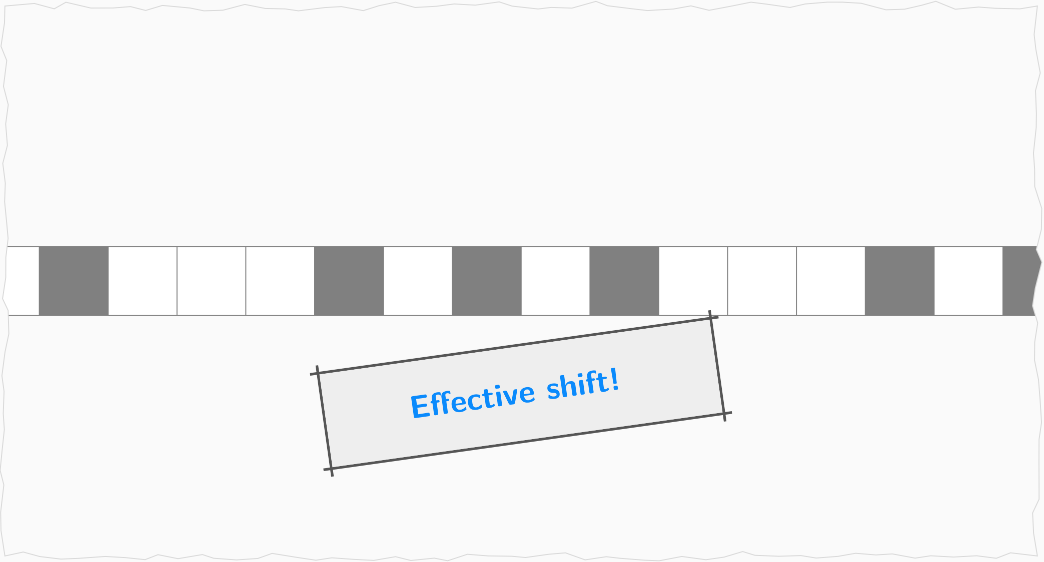
$h = .10100\ldots$

What about topological entropies? (Part 2)



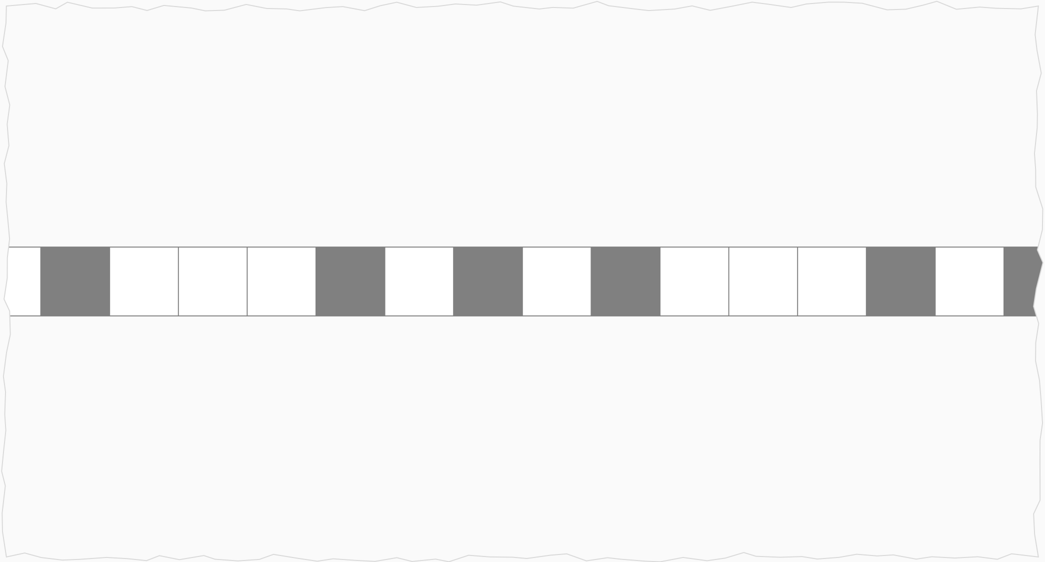
$$h = .10100\ldots$$

What about topological entropies? (Part 2)



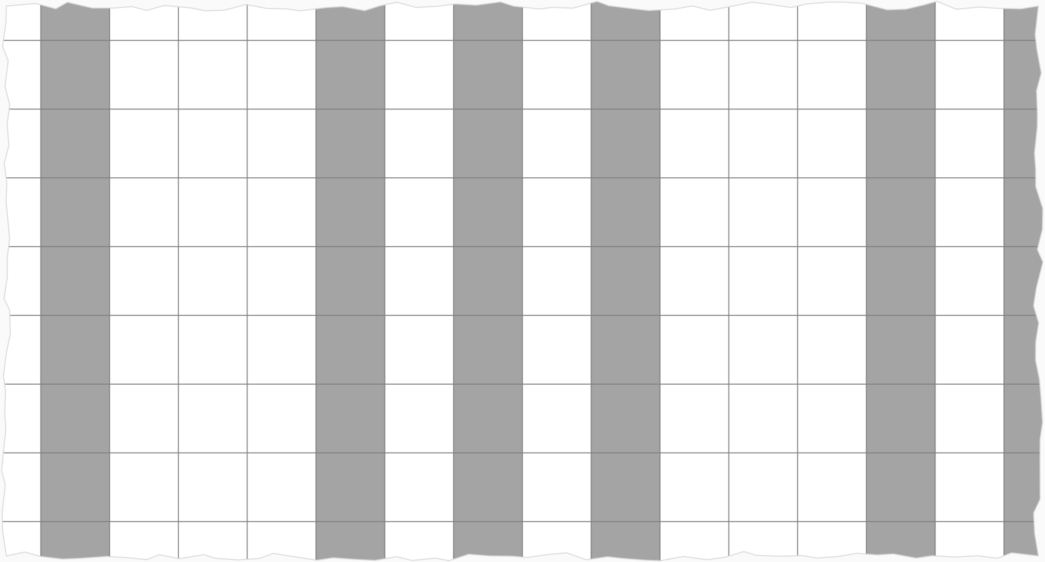
$$h = .10100\ldots$$

What about topological entropies? (Part 2)



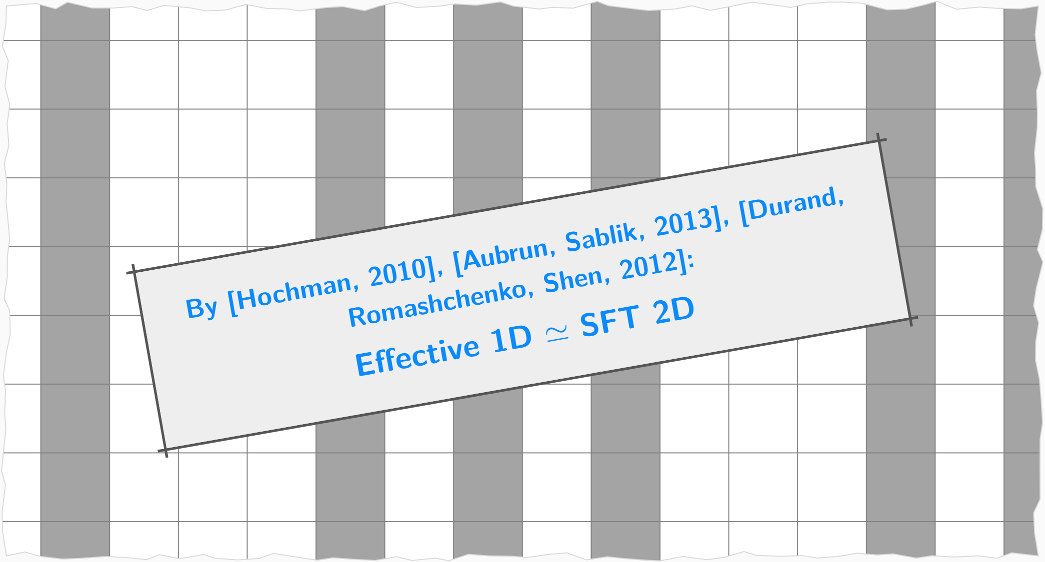
$h = .10100\ldots$

What about topological entropies? (Part 2)



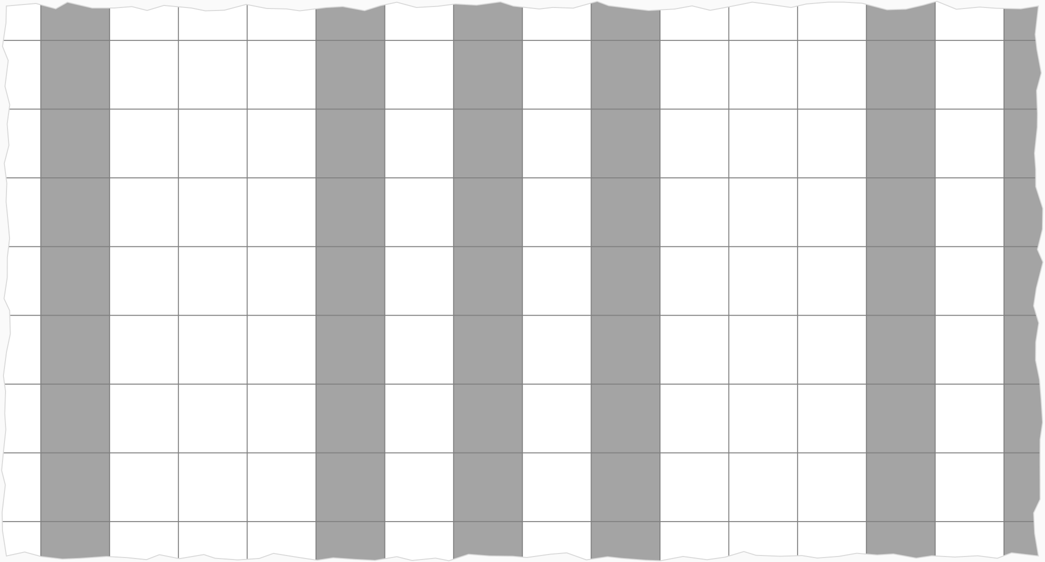
$$h = .10100...$$

What about topological entropies? (Part 2)



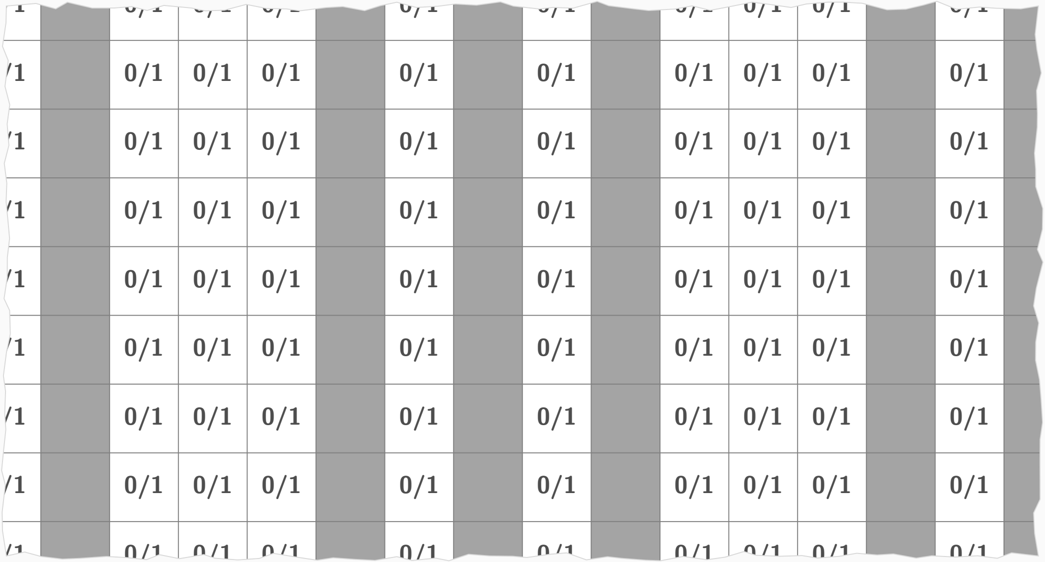
$h = .10100\ldots$

What about topological entropies? (Part 2)



$h = .10100...$

What about topological entropies? (Part 2)



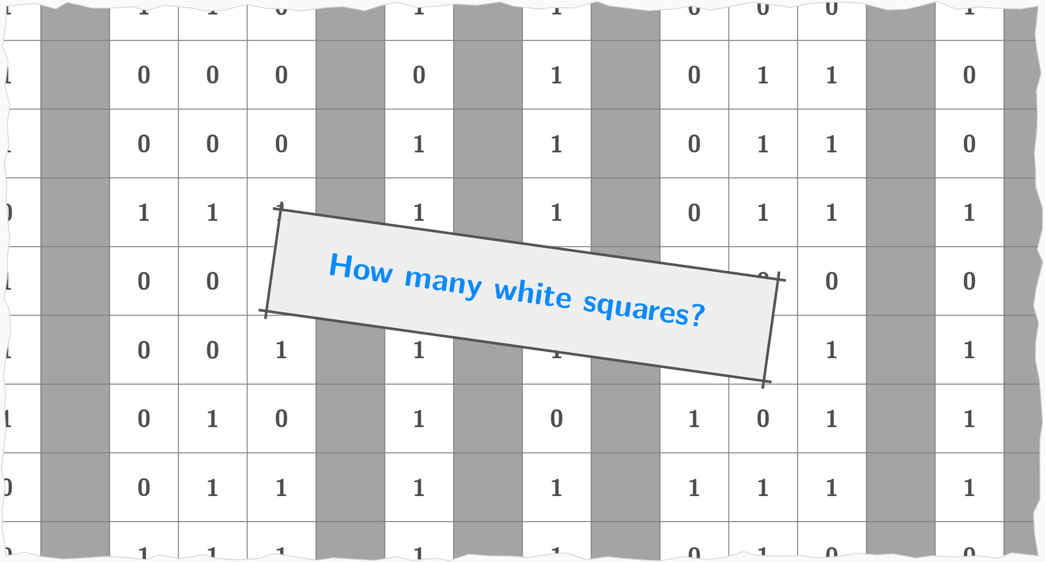
$$h = .10100...$$

What about topological entropies? (Part 2)

0	0	0	0	1	0	1	1	0
0	0	0	1	1	0	1	1	0
1	1	1	1	1	0	1	1	1
0	0	0	0	1	0	0	0	0
0	0	1	1	1	1	0	1	1
0	1	0	1	0	1	0	1	1
0	1	1	1	1	1	1	1	1

$$h = .10100\ldots$$

What about topological entropies? (Part 2)



$$h = .10100\ldots$$

What about topological entropies? (Part 2)

1		1	1	0		1		1		0	0	0		1	
1		0	0	0		0		1		0	1	1		0	
1		0	0	0		1		1		0	1	1		0	
0		1	1	1		1		1		0	1	1		1	
1		0	0	0		0		1		0	0	0		0	
1		0	0	1		1		1		1	0	1		1	
1		0	1	0		1		0		1	0	1		1	
0		0	1	1		1		1		1	1	1		1	
0		1	1	1		1		1		0	1	0		0	

$$h = .10100\ldots$$

What about topological entropies? (Part 3)

$$\log N_n \simeq hn^2 \qquad \Longrightarrow \qquad h_{\mathsf{top}} = h$$

Surface entropy

Surface entropies (Part 1)

$$\log N_n \simeq hn^2 + h'n$$

$\Longrightarrow$: Surface entropies are Π_3 real numbers.

$\Longleftarrow$: Let $h' \in \Pi_3$,

$$h' = \limsup_{k \to +\infty} r_k$$

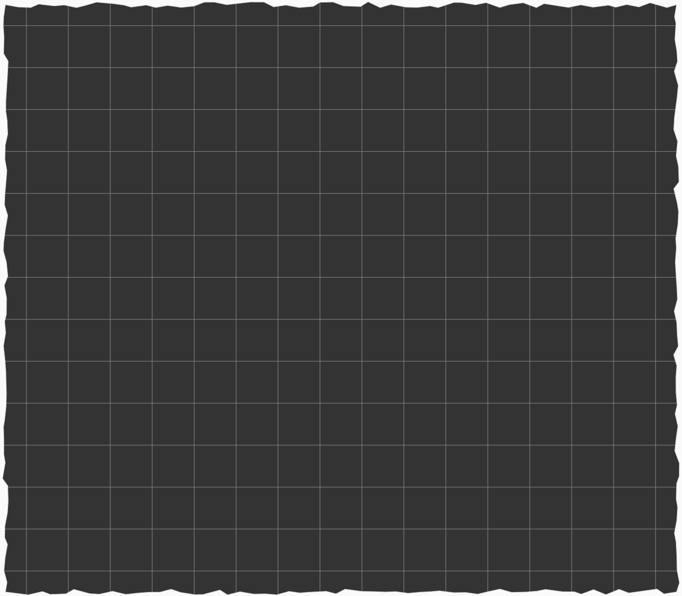
We create an SFT with surface entropy h'.

Arithmetical hierarchy of real numbers

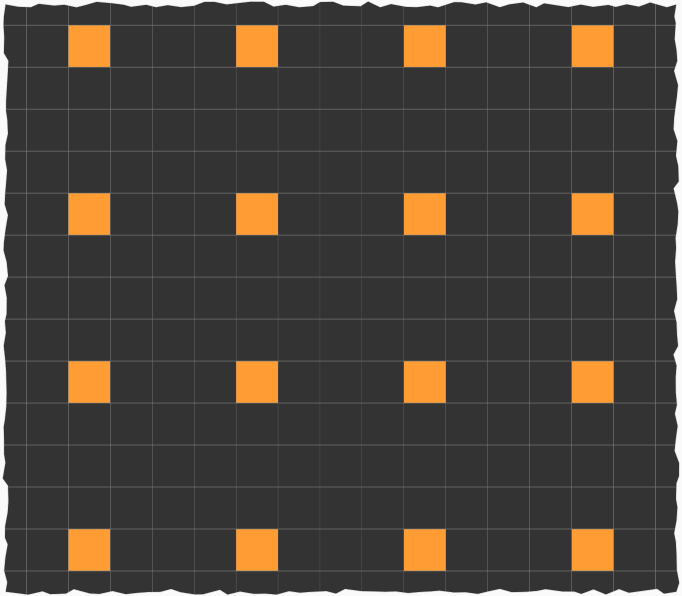
$x \in \Pi_3$ if there exists a recursively enumerable $(r_k)_{k \in \mathbb{N}}$ in Π_1 such that:

$$x = \limsup_k r_k$$

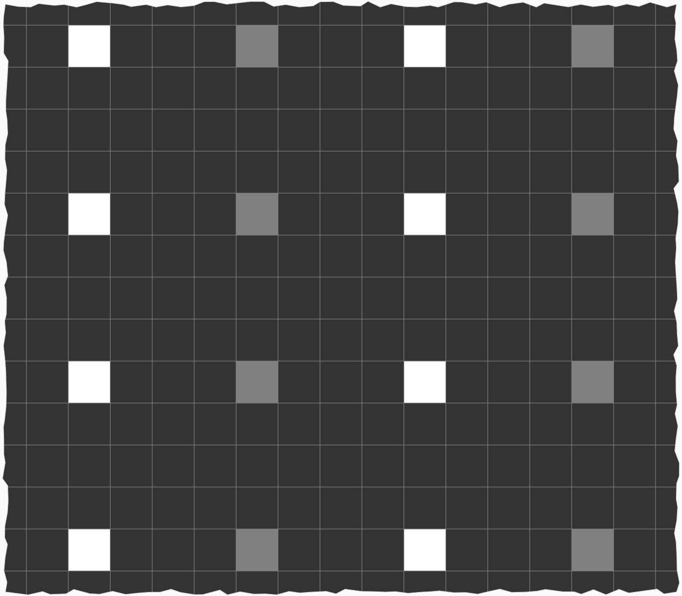
Surface entropies: the sparse squares (Part 2)



Surface entropies: the sparse squares (Part 2)

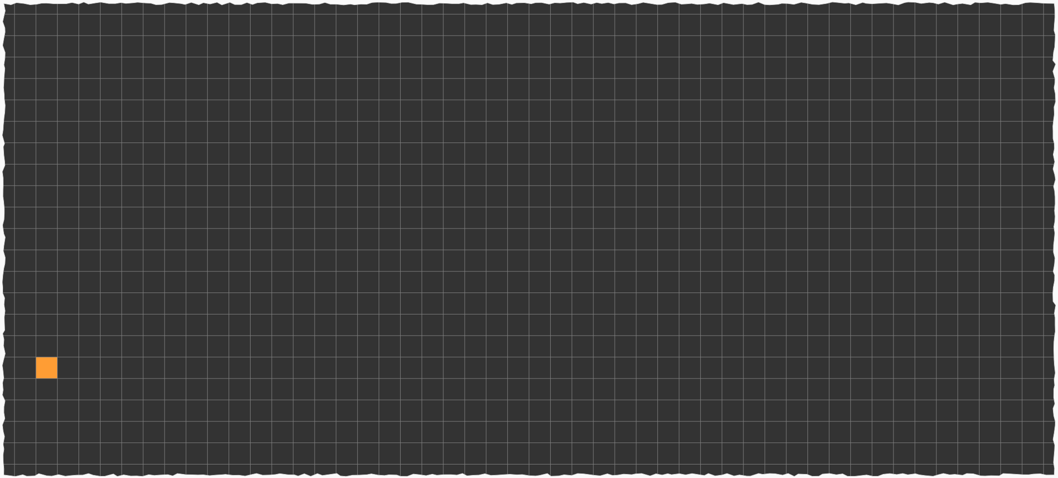


Surface entropies: the sparse squares (Part 2)

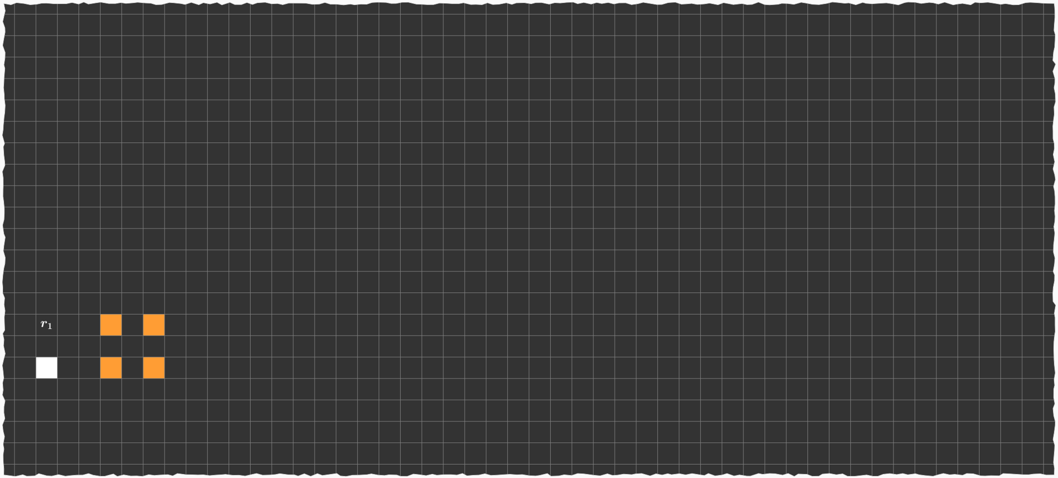


$r_4 = 1/2$

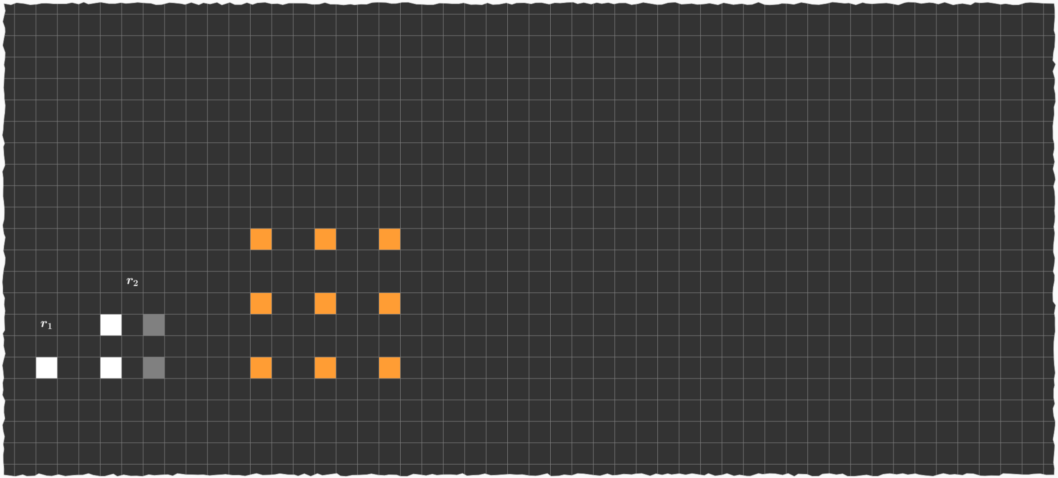
Surface entropies: structure (Part 3)



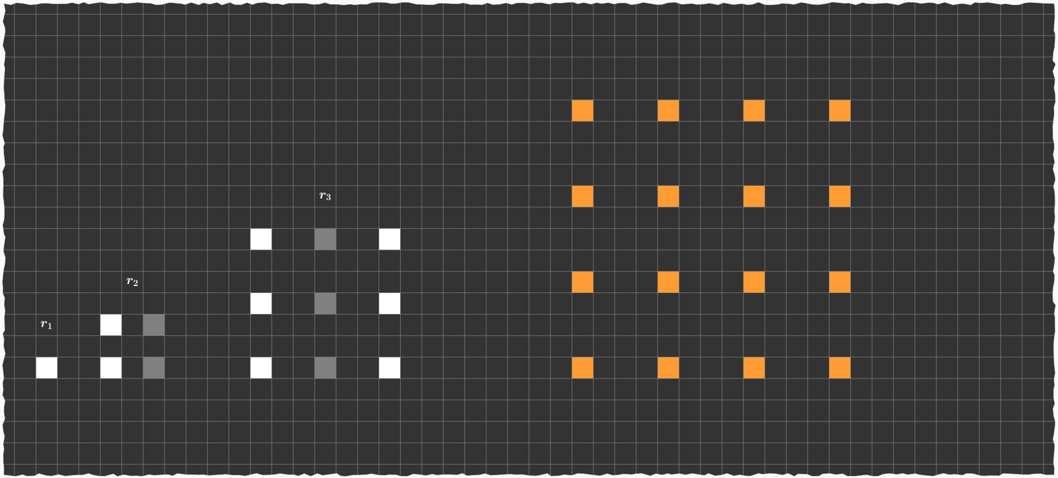
Surface entropies: structure (Part 3)



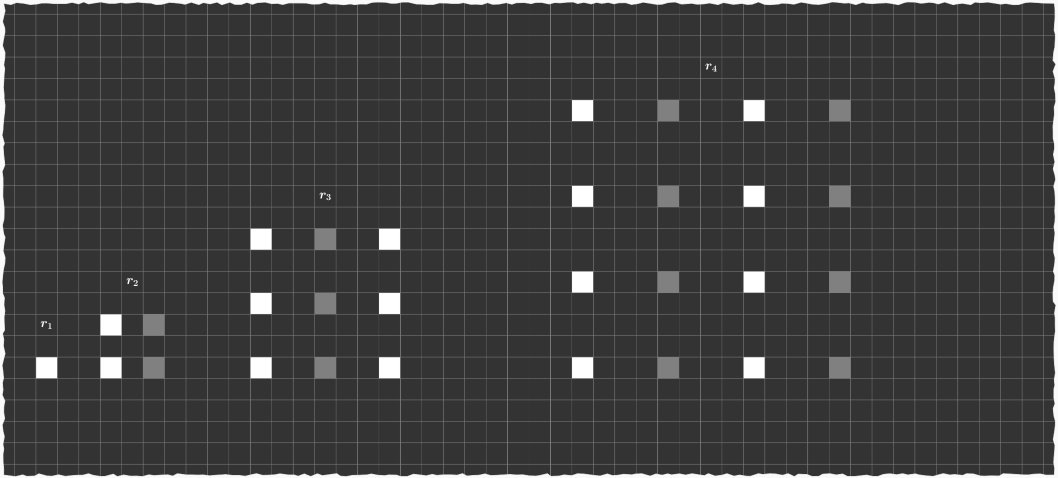
Surface entropies: structure (Part 3)



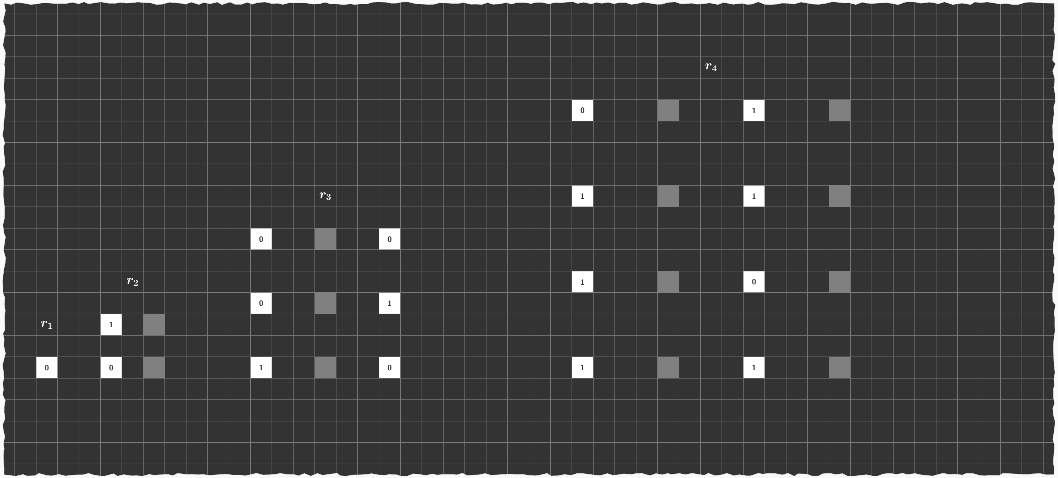
Surface entropies: structure (Part 3)



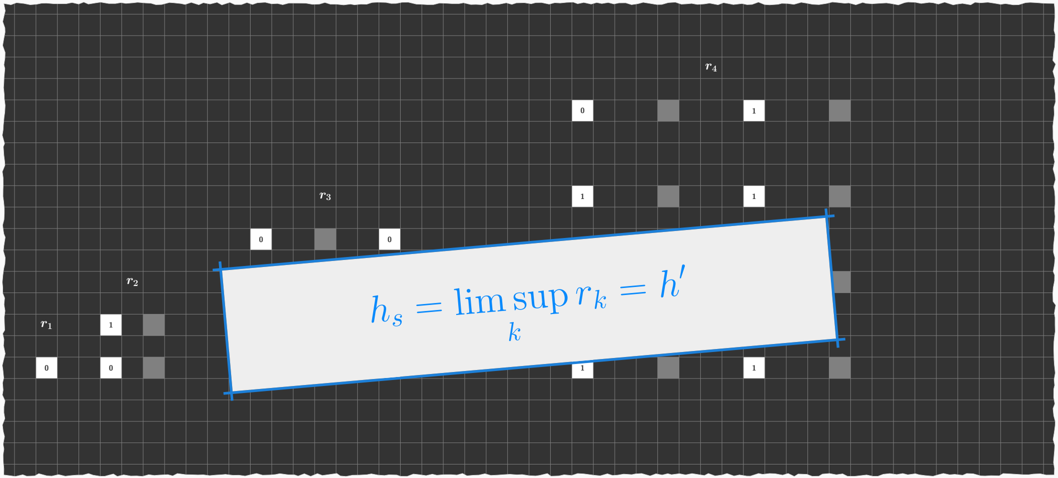
Surface entropies: structure (Part 3)



Surface entropies: structure (Part 3)

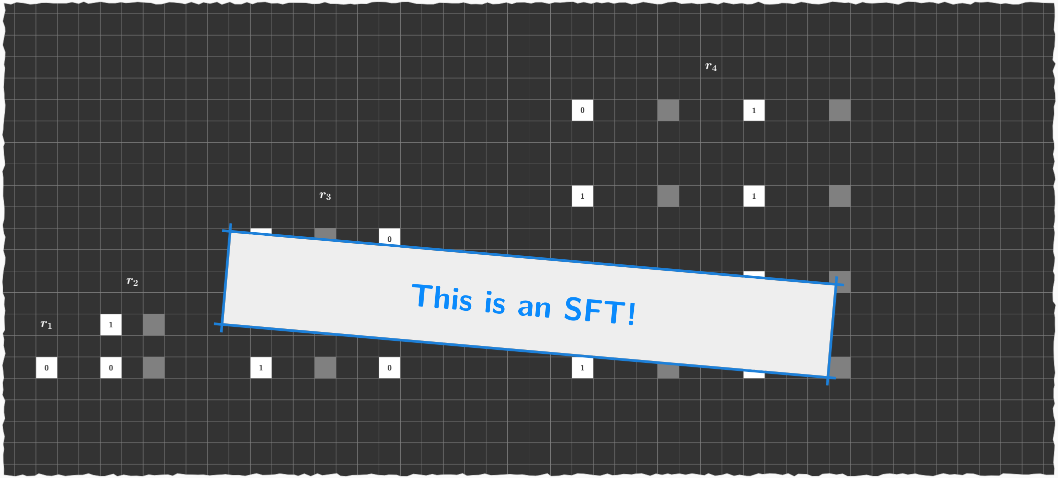


Surface entropies: structure (Part 3)

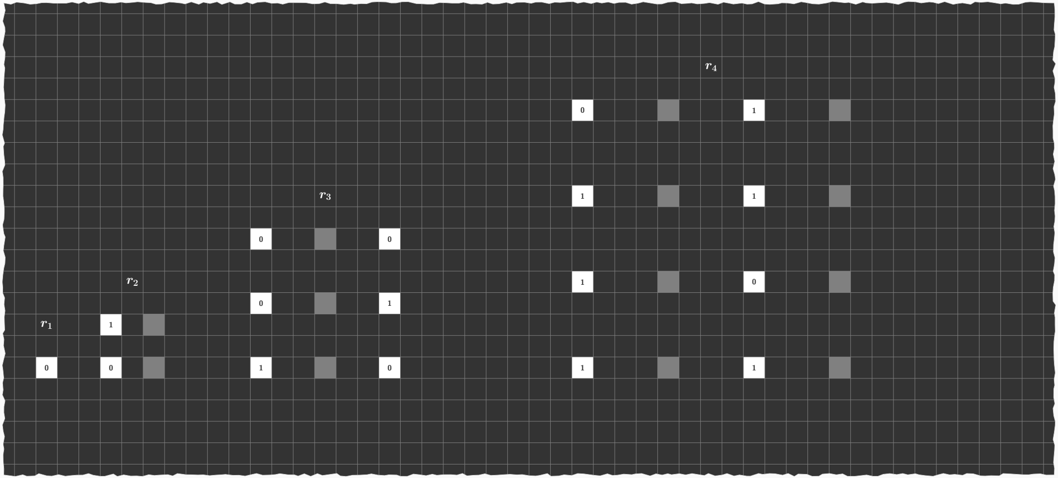


$$h_s = \limsup_k r_k = h'$$

Surface entropies: structure (Part 3)



Surface entropies: structure (Part 3)



Conclusion

Conclusion

Question:

$$\log N_n \simeq hn^2 + h'n$$

In an SFT, what are the possible values for h' ?

Answer:

Surface entropies are exactly the class of Π_3 real numbers!

Questions?